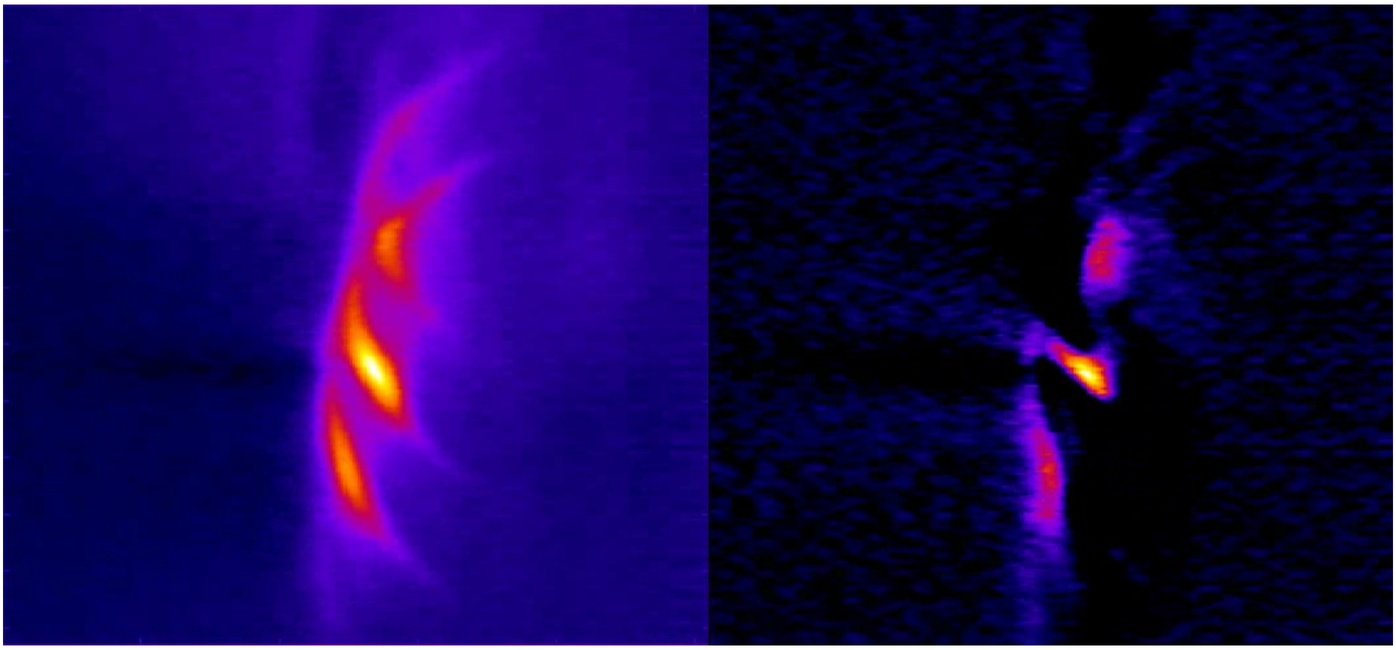
LINEAR FLUID MODEL FOR DRIFT INSTABILITIES IN A MAGNETIZED PLASMA

(09/2025)



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We study the drift wave: this is an electrostatic instability linked to the presence of an electron density gradient inside a magnetized plasma.

We first introduce the plasma diamagnetic drift due to density or temperature gradients.

Using a basic linear fluid model, we study how this instability can propagate in the plasma in the direction perpendicular to the magnetic field and to the density gradient. The propagation phase velocity involves the electron diamagnetic velocity.

In order to describe the unstable nature of the mode, the model must include a phase shift between the potential and the electron density. We study the effect of this phase shift on the fluctuation behavior. The physical meaning of this phase shift is briefly introduced.

The derivative notations are simplified. For the spatial partial derivative: $\partial_x = \frac{\partial}{\partial x}$,

and for the time derivative: $d_t = \frac{d}{dt}$.

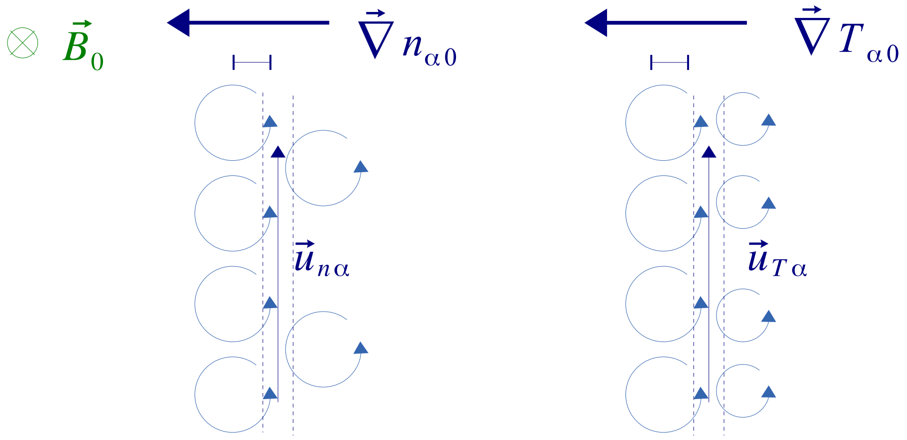
Contents

1 The Diamagnetic Drift velocity

We introduce the diamagnetic drift due to density or temperature gradients inside a magnetized plasma.

1.1 The graphical description

The diamagnetic drift differs from the other main drifts in magnetized plasmas (ExB drift, or magnetic B field curvature or gradient drift): it does not modify the particle cyclotron motion: the particle guiding centers still follow the magnetic field lines. The diamagnetic drift is the result of a particle velocities imbalance on the scale of the Larmor radius, for a given position.



In the presence of a species density gradient $\vec{\nabla} n_{\alpha 0}$, as the density of particle guiding centers is not uniform around a given position, the particle mean velocity $\vec{u}_{n\alpha}$ at this position is non-zero:

this mean velocity is along the direction perpendicular to the magnetic field and to the density gradient. This mean velocity direction depends on the most numerous particles: those whose guiding center is located where the particle density is higher.

In the presence of a particle species temperature gradient $\vec{\nabla} T_{\alpha 0}$, as the mean thermal velocity is different for the guiding centers around a given position, the particle species mean velocity $\vec{u}_{T\alpha}$ is non-zero: the direction of the mean velocity corresponds to that of the particles whose guiding center is on the side of the largest thermal velocities.

This drift velocity depends on the species thermal velocity, $u_{T\alpha}$, on its mean Larmor radius $\rho_{cT\alpha}$ and on the pressure gradient.

1.2 The fluid model approach

In order to introduce this diamagnetic drift velocity, we use a fluid description of a magnetized plasma equilibrium with a pressure gradient (whether it is linked to a density gradient or a temperature gradient).

The species α momentum conservation equation, for a magnetized plasma with no electric field, is:

$$n_\alpha m_\alpha\left(\frac{\partial \vec{u}_\alpha}{\partial t}+\vec{u}_\alpha.\nabla \vec{u}_\alpha\right)=n_\alpha q_\alpha \vec{u}_\alpha\times\vec{B}_0-\vec{\nabla} P_\alpha$$

For a stationary state, the expression simplifies at the force balance:

$$n_\alpha q_\alpha \vec{u}_\alpha\times\vec{B}_0-\vec{\nabla} P_\alpha=\vec{0}$$

This equation does not depend on the parallel part of the fluid velocity. The parallel velocity does not the pressure gradient : the parallel part of the diamagnetic velocity is null:

$$\vec{u}_{\alpha\parallel}=\vec{0}$$

The relation implies that the pressure gradient should be perpendicular to the magnetic field.

$$\vec{B}_0\perp\vec{\nabla} P_\alpha$$

if we multiply both parts of the equation by $\times\vec{B}$ and using:

$$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$$

The particle velocity is:

$$\vec{u}_{P\alpha}=\frac{-1}{n_\alpha q_\alpha B_0^2}\vec{\nabla} P_\alpha\times\vec{B}_0$$

The pressure expression is:

$$P_\alpha=\gamma_\alpha n_\alpha k_B T_\alpha$$

with, for the electrons, $\gamma_e=1$, and for monatomic ions, $\gamma_i=3$, this velocity is:

$$\vec{u}_{P\alpha}=\frac{-\gamma_\alpha k_B T_\alpha}{q_\alpha B_0}\frac{\vec{\nabla} P_\alpha}{P_\alpha}\times\frac{\vec{B}_0}{B_0}$$

It can also be expressed using particle mean parameters :

$$\vec{u}_{P\alpha}=-u_{T\alpha}\rho_{cT\alpha}\frac{\vec{\nabla} P_\alpha}{P_\alpha}\times\frac{\vec{B}_0}{B_0} \qquad (1.1)$$

where :

$$u_{T\alpha}=\sqrt{\frac{\gamma_\alpha k_B T_\alpha}{m_\alpha}}\ ,$$

$$\rho_{cT\alpha}=\frac{u_{T\alpha}}{\omega_{c\alpha}}$$

and

$$\omega_{c\alpha}=\frac{q_\alpha B_0}{m_\alpha}\ .$$

This velocity is in the direction perpendicular to the magnetic field and the pressure gradient as described graphically.

2.a Gradient length

The velocity expression introduces a length commonly called the gradient length.

For any 1D function of a position $f(x)$, the gradient length is defined as :

$$l_x=\frac{f}{d_x f}$$

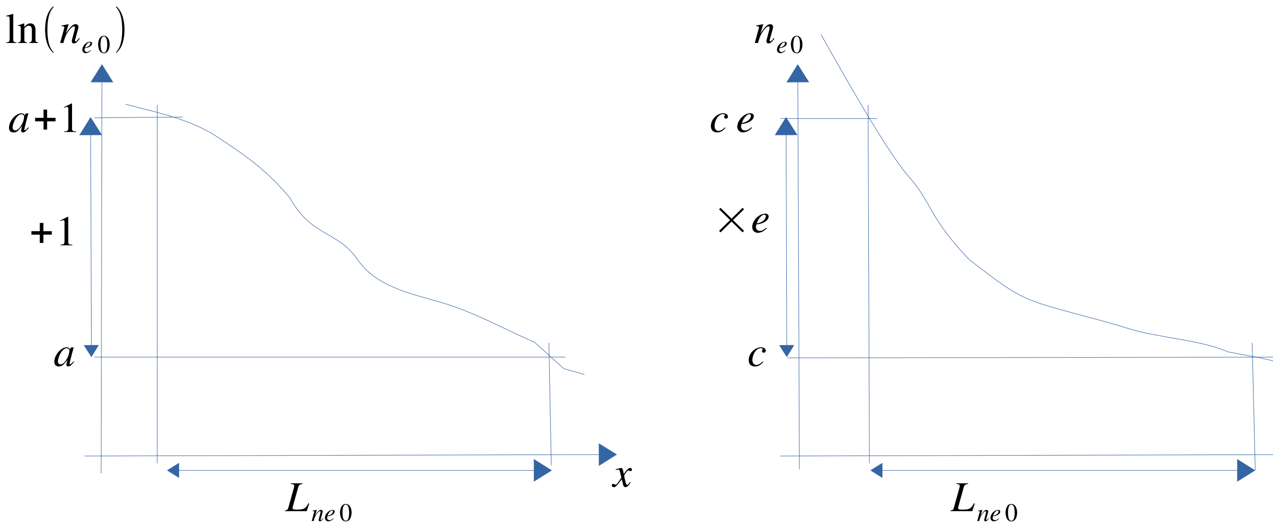
or :

$$l_x = \frac{1}{d_x \ln f}$$

The gradient length can be estimated by the logarithm of the function, by estimating over what length the function logarithm varies by addend 1 . It can also be estimated directly on the graph of the function, by estimating over which length the density varies by a factor e (i.e. about 2.7).

For a 3D field parameter, such as pressure, the gradient length becomes :

$$l_{P\alpha} = \frac{P_\alpha}{|\nabla P_\alpha|}$$



The steeper the gradient is, the shorter the gradient length is.

Diamagnetic velocity expression

The diamagnetic velocity modulus has the form :

$$u_{P\alpha} = \frac{\gamma_\alpha k_B T_\alpha}{q_\alpha B_0 l_{P\alpha}} \qquad (1.2)$$

The diamagnetic velocity is larger for the more energetic particles.

Another equivalent expression is

$$u_{P\alpha} = \frac{\rho_{cth\alpha}}{l_{P\alpha}} u_{th\alpha}$$

The pressure gradient length is generally much larger than the particle Larmor radius (especially for electrons): the diamagnetic velocity is much lower than the thermal velocity of the particles.

1.3 The kinetic description

The fluid description is simple and brings out the diamagnetic velocity. but it is artificial : it needs a equilibrium assumption. The diamagnetic velocity can be directly deduced from the particle velocity distribution : this is the kinetic description.

The guiding center distribution

This is a second approach to evaluate the diamagnetic velocity with a plasma kinetic description. This evaluation is more complex, but it needs no equilibrium assumption.

The particle behavior depends on the particle guiding center invariant position, X . The guiding center position and the particle position x are linked by the particle velocity as a consequence of the particle cyclotron motion:

$X = x + \frac{v_y}{\omega_{c\alpha}}$

Axial guiding center density profiles $n_{g\alpha}(X)$ and temperature profiles $T_\alpha(X)$ are steady state.

We note $f_{g\alpha}(X, \vec{v})$ the particle distribution function, expressed as a function of the position of the guide center :

$$f_{g\alpha}(X, \vec{v}) = \left(\frac{m_\alpha}{2\pi k_B T_\alpha(X)} \right)^{3/2} n_{g\alpha}(X) e^{\frac{-m_\alpha v^2}{2 k_B T_\alpha(X)}} \quad (1.3)$$

Around the guide center, the distribution of particle velocities is Maxwellian, with temperature $T_\alpha(X)$.

We assume the characteristic gradient length for the density $L_{n\alpha} = \langle d_x \ln(n_{0\alpha}) \rangle^{-1}$ and for the temperature $L_{T\alpha} = \langle d_x \ln(T_\alpha) \rangle^{-1}$ are long compared to the Larmor radius (i.e. the gradients are small) :

$\rho_{c\alpha} L_{n\alpha} \ll 1$ et $\rho_{c\alpha} L_{T\alpha} \ll 1$

The particle distribution function $f_\alpha(x, H)$ expressed as a function of the particle position x depends on the guiding center distribution function and the cyclotron motion around the guiding center:

$f_{0\alpha}(x, \vec{v}) = f_{g\alpha}(x + \frac{v_y}{\omega_{c\alpha}}, \vec{v})$

The value is approximated by a Taylor series, from the guiding center distribution function.

The particle distribution function as a function of the particle position is :

$$f_{0\alpha}(x, \vec{v}) = f_{g\alpha}(x, \vec{v}) + \partial_X f_{g\alpha}(x, \vec{v}) \frac{v_y}{\omega_{c\alpha}} \quad (1.4)$$

The relation between the particle and guiding center distribution functions depends on the density and temperature gradients:

$$\partial_x f_{g\alpha}(x, \vec{v}) = \left[d_X \ln(n_{g\alpha}) + d_X \ln(T_\alpha) \left(\frac{\frac{1}{2} m_\alpha v^2}{k_B T_\alpha} - \frac{3}{2} \right) \right] f_{g\alpha}(x, \vec{v}) \quad (1.5)$$

Because of these gradients, the particle distribution function is no longer Maxwellian.

Particle density

The species α density is :

$n_{0\alpha}(x) = \int d^3\vec{v}\, f_{0\alpha}(x, H)$

or :

$n_{0\alpha}(x) = \int d^3\vec{v}\, f_{g\alpha}(x, \vec{v}) + \int d^3\vec{v}\, \partial_x f_{g\alpha}(x, \vec{v}) \frac{v_y}{\omega_{c\alpha}}$

Since the second integrated function is antisymmetric in v_y , the second term is zero.

The particle density is identical to the guiding center density profile:

$$\boxed{n_{0\alpha}(x)=n_{g\alpha}(x)} \tag{1.6}$$

Mean drift velocity

Because of the density or temperature gradients, the species α mean velocity at any position is non zero : this is the diamagnetic drift.

$$\langle \vec{v_{0\alpha}}(x)\rangle=\frac{1}{n_{0\alpha}}\int d^3\vec{v}\,\vec{v}\,f_{0\alpha}(x,H)$$

or :

$$\langle \vec{v_{0\alpha}}(x)\rangle=\frac{1}{n_{0\alpha}}\int d^3\vec{v}\,\vec{v}\,f_{g\alpha}(X,\vec{v})+\frac{1}{n_{0\alpha}}\int d^3\vec{v}\,\vec{v}\,\partial_X f_{g\alpha}(X,\vec{v})\frac{v_y}{\omega_{c\alpha}}$$

Because the first integrated function is antisymmetric with respect to the speed, the first term is zero. For the second term, the function under the integral following the directions $\vec{e}_x$ and $\vec{e}_z$ is also antisymmetric. It remains :

$$\langle \vec{v_{0\alpha}}(x)\rangle=\frac{1}{n_{0\alpha}}\int d\,v_y\left[d_X\ln(n_{g\alpha})+d_X\ln(T_\alpha)\left(\frac{\frac{1}{2}m_\alpha v^2}{k_B T_\alpha}-\frac{3}{2}\right)\right]f_{g\alpha}(x,\vec{v})\frac{v_y^2}{\omega_{c\alpha}}\vec{e}_y$$

since for a Maxwellian:

$$\frac{1}{n_\alpha}\int d\,v_y v_y^2 f_\alpha=v_{th\alpha}^2 \quad \text{et} \quad \frac{1}{n_\alpha}\int d\,v_y v_y^4 f_\alpha=3\,v_{th\alpha}^4$$

the mean velocity is :

$$\langle \vec{v_{0\alpha}}(x)\rangle=\left[d_X\ln(n_{0\alpha})+d_X\ln(T_\alpha)\right]\frac{k_B T_\alpha}{q_\alpha B}\vec{e}_y$$

Since the pressure for the species α is :

$$P_\alpha=n_{0\alpha}k_B T_\alpha$$

Its gradient is:

$$\nabla P_\alpha=n_{0\alpha}k_B T_\alpha\left(d_X\ln(n_{0\alpha})+d_X\ln(T_\alpha)\right)\vec{e}_x$$

The drift velocity is a function of the pressure gradient :

$$\boxed{\langle \vec{v_{0\alpha}}(x)\rangle=\frac{-\nabla P_\alpha\wedge\vec{B}_0}{q_\alpha n_{0\alpha}B_0^2}} \tag{1.7}$$

This velocity, defined from the pressure gradient, is the diamagnetic drift of the species α . Unlike other drifts ($\vec{E}_0\wedge\vec{B}_0$ or $\vec{\nabla} B_0\wedge\vec{B}_0$) , this drift does not concern the guiding center trajectory, but is a collective effect.

The sign depends on the charge sign: the ionic and electronic diamagnetic drifts have opposite signs.

2 The simplified drift wave linear fluid model

2.1 Instability linear fluid model description

1.a Electrostatic instability in a magnetized plasma

We study the electrostatic drift instability in magnetized plasmas. We simplify the geometry to a "slab" reference frame : a 3D Cartesian reference frame $(\vec{e}_x, \vec{e}_y, \vec{e}_\phi)$ in which the plasma parameter mean values vary only along 1 direction (here $\vec{e}_x$).

The magnetic field is uniform, along $\vec{e}_\phi$:

$\vec{B} = B_{\phi 0} \vec{e}_\phi$.

The magnetic field magnitude is of the order of $0{,}1\, T$.

The magnetic field curvature and gradient, always present in toric plasmas, are not taken into account in this very simplified model.

The plasma, created from a noble mono-atomic gas (here Argon) is a cold plasma : it is only partially ionized and constituted with 2 charged species : the electrons e^- , and only one positive ion species, i^+ .

The electron temperature is of the order of $1\, eV$, The ions remain at the ambient temperature. The electron density remains below $10^{17} m^{-3}$. In these conditions, the plasma does not generate an intrinsic magnetic field.

We consider only electrostatic instabilities : the magnetic field remains constant. The plasma potential φ_1 varies. The electric field is only function of the electric potential :

$\vec{E}_1 = -\vec{\nabla} \varphi_1$

1.b Perturbative description

We use a perturbative description to study this instability. We consider the instability is only a weak perturbation on the plasma density.

First order development

The electron density is the sum of steady state mean part, and the fluctuating part:

$n_e(\vec{r}, t) = n_{e0}(\vec{r}) + n_{e1}(\vec{r}, t)$

The time mean fluctuating part is null :

$\langle n_{e1}(\vec{r}, t) \rangle_t = 0$

The fluctuating part standard deviation is much smaller than the mean value :

$\sqrt{\langle n_{e1}(\vec{r}, t)^2 \rangle_t} \ll n_{e0}(\vec{r})$

Equation linearization

The equation describing the plasma dynamics is approximated with limited Taylor development with the perturbation.

The development 0th order corresponds to the steady state plasma with spatial variations but with no time fluctuations.

We assume the perturbation is small compared to the time mean values. We introduces the development 1st order. Once the 0th order is subtracted from the 1st order equations, the linear terms of the equation only remains.

1.c A single Fourier mode

Due to the plasma dynamics equations, The eigenvalues of such equation systems are Fourier modes. The eigenvalues are monochromatic electrostatic waves with the wave vector $\vec{k}$ and the angular frequency ω . For the simplest cases, the perturbation wave vector is perpendicular to the magnetic field lines (along $\vec{e}_z$) to the plasma potential and electron density gradients (here along $\vec{e}_x$) :

$$\vec{k}=k_y\vec{e}_y$$

In order to simplify the expressions, we use the wave complex notation. All fluctuating parameters, for the 1st order, have an expression similar to the electron density one :

$$n_{e1}(\vec{r},t)=\tilde{n}_{e1}e^{i(k_y y-\omega t)}$$

For a single Fourier mode, the fluctuation spatial gradient has a linear expression:

$$\vec{\nabla}n_{e1}=i\,k_y n_{e1}\vec{e}_y$$

The fluctuation time derivative also has a linear expression:

$$\frac{\partial}{\partial t}n_{e1}=-i\,\omega\,n_{e1}$$

Fluctuation growth rate

For more complex linear models, the fluctuation angular frequency may include an imaginary part :

$$\omega=\omega_R+i\,\gamma$$

The fluctuation expression is more complete:

$$n_{e1}(\vec{r},t)=\tilde{n}_{e1}e^{\gamma t}e^{i(k_y y-\omega_R t)}$$

In this case, the mode magnitude varies exponentially with time :

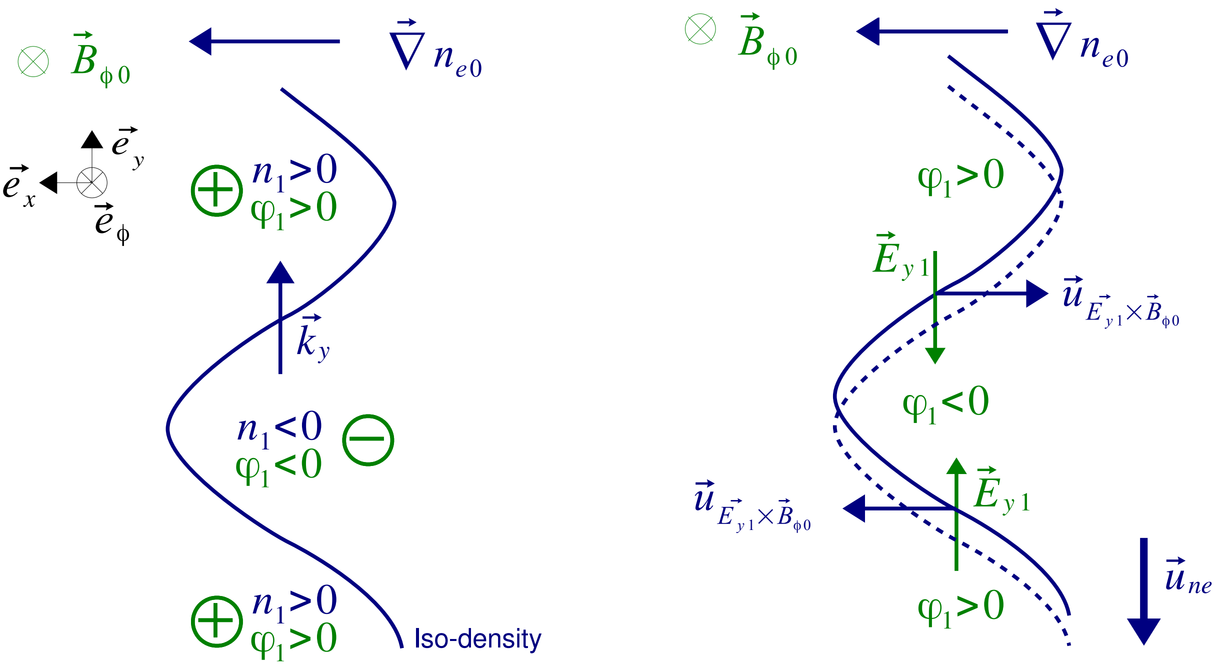
$$|n_{e1}(\vec{r},t)|=\tilde{n}_{e1}e^{\gamma t}$$

The angular frequency imaginary part, γ , is the wave magnitude growth rate. The growth rate sign determines the wave stability nature. If γ est positive, the wave magnitude increases exponentially : the mode is unstable. If γ est negative, the magnitude decreases exponentially.

2.2 The propagation model

We restrict the study to a drift mode linked to a density gradient. The ion temperature T_i and the electron temperature T_e is uniform.

2.a The graphical description of the propagation



The magnetic field $\vec{B}$ is along $\vec{e}_\phi$: $\vec{B}_{\phi 0}$.

The electron density gradient $\vec{\nabla} n_{e0}$ is along $\vec{e}_x$.

The electron density perturbation is along $\vec{e}_y$, perpendicular to the magnetic field and the electron density gradient.

The perturbation distorts the plasma iso-density curves. Density bumps ($n_{e1}>0$) correspond to over-densities and dips ($n_{e1}<0$) to low densities along the perturbation.

Since the plasma potential variations have the same sign as the electron density variations (because of the balance between the pressure and electric forces on the electron), the potential is positive ($\varphi_1>0$) in the electron density bumps, and negative ($\varphi_1<0$) in the dips.

These plasma potential variations induce the formation of an electric field $\vec{E}_{y1}$ along the perturbation mode wave vector $\vec{k}_y$ direction. The field $\vec{E}_{y1}$ orientation reverses between density bumps and dips. Because of the the magnetic field, this electric field induces a plasma drift velocity $\vec{u}_{\vec{E}_{y1}\times\vec{B}_{\phi 0}}$, in the direction of the density gradient. This drift is also alternated along $\vec{k}_y$ between the bumps and the dips.

This alternating drift orthogonal motion along the perturbation results in the iso-density curve displacement along $\vec{k}_y$, in the direction of the electron diamagnetic velocity $\vec{u}_{ne}$: the mode propagates.

2.b The linear fluid model

A basic fluid model helps to obtain a drift mode propagation velocity expression.

The density gradient $\vec{\nabla} n_{e0}$ is along $\vec{e}_x$. The gradient magnitude is $\partial_x n_{e0}$.

Electrons are characterized by their density n_e , their temperature T_e (the temperature variations are neglected), the plasma potential is φ . The ions have the same density as electrons. In this simplified model, only ion velocity $\vec{u}_i$ differs from electrons.

The instability is described as a linear (1st order) perturbation of the equilibrium. The parameters are written as the sum of their equilibrium value, plus fluctuations:

$n_e(\vec{r},t)=n_{e0}(\vec{r})+n_{e1}(\vec{r},t)$

The fluctuations are restricted to one mode along $\vec{e}_y$:

$n_{e1}(\vec{r},t)=\tilde{n}_{e1}e^{i(k_y y-\omega t)}$

In order to describe the wave propagation, we first describe the electron dynamics. The wave angular frequency ω is small compared to the electron plasma frequency, ω_{pe} : in the equation on the electron dynamics, their inertia ($m_e d_t \vec{u}_e$) is neglected. At any time, locally, the electrons reach a Boltzmann equilibrium. This equilibrium assumes that the electrons circulate freely between the low potential energy zones and the high potential energy zones, despite the magnetic confinement.

The electron density depends only on the electric potential energy ($q_e>0$ is the elementary charge) :

$n_{e0}+n_{e1}=n_{e0}e^{\frac{q_e\varphi_1}{k_B T_e}}$

The perturbative development first order is:

$n_{e1}=\frac{q_e\varphi_1}{k_B T_e}n_{e0}$

or :

$$\varphi_1=\frac{k_B T_e}{q_e n_{e0}}n_{e1} \qquad (2.1)$$

The potential varies like the density, along $\vec{e}_y$. The electric field $\vec{E}_1$ linked to this potential is also along $\vec{e}_y$. In the presence of the magnetic field $\vec{B}_{\phi 0}$, the electric field $\vec{E}_1$ induces the ion oscillating motion due to the ExB drift $\vec{u}_{\vec{E}_{y1}\times\vec{B}_0}$:

$\vec{u}_{i1}=\frac{\vec{E}_1\times\vec{B}_{\phi 0}}{B_{\phi 0}^2}$

this drift is along $\vec{e}_x$:

$u_{ix1}=-i k_y\frac{\varphi_1}{B_{\phi 0}}$

For the ion dynamics, we consider the continuity equation :

$\partial_t n_i+\vec{\nabla}.(n_i\vec{u}_i)=0$

At the 1st order, the ion continuity equation is :

$-i\omega n_{e1}+u_{ix1}\partial_x n_{e0}=0$

We take into account the expression found for the ion velocity:

$$\omega n_{e1}=-k_y\frac{\varphi_1}{B_{\phi 0}}\partial_x n_{e0} \qquad (2.2)$$

We obtain 2 different linear equations between the mode parameters φ_1 and n_{e1} , (2.1) and (2.2). They are multiplied:

$$\varphi_1 \omega n_{e1} = -k_y \frac{k_B T_e \partial_x n_{e0}}{q_e B_{\phi 0} n_{e0}} \varphi_1 n_{e1}$$

The solutions with non zero φ_1 and n_{e1} , must satisfy the equation:

$$\omega = -k_y \frac{k_B T_e}{q_e B_{\phi 0}} \frac{\partial_x n_{e0}}{n_{e0}}$$

This is the wave dispersion relation, linking the angular frequency ω to the wave number k_y . Their ratio corresponds to the mode propagation velocity.

$$u_\phi = \frac{\omega}{k_y} = -\frac{k_B T_e}{q_e B_{\phi 0}} \frac{\partial_x n_{e0}}{n_{e0}} \tag{2.3}$$

This velocity is the electron diamagnetic velocity, in the case of a density gradient, with no temperature gradient ($q_e > 0$ elementary charge) :

$$u_{ne} = \frac{-k_B T_e}{q_e B_{\phi 0}} \frac{\partial_x n_{e0}}{n_{e0}}$$

The vector expression for the phase velocity is:

$$\vec{u}_\phi = \vec{u}_{ne} = \frac{k_B T_e}{n_e q_e B_{\phi 0}^2} \vec{\nabla} n_e \times \vec{B}_{\phi 0} \tag{2.4}$$

If in the plasma, a mean electric field on the volume $\vec{E}_0$ is also present, all the plasma is convected by the global ExB velocity:

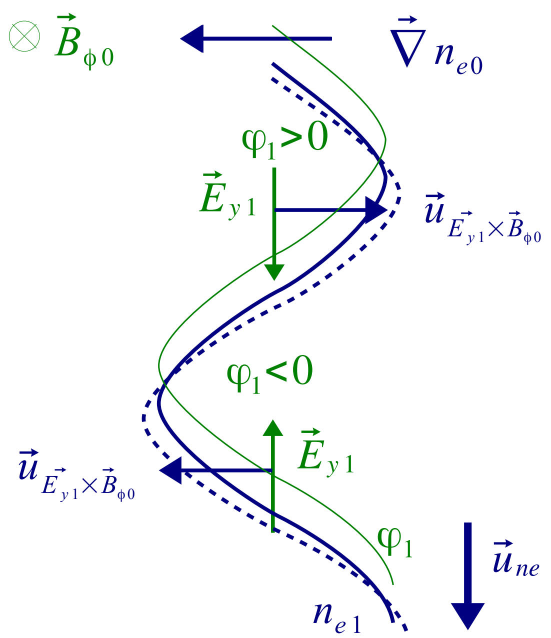
$$\vec{u}_{ExB0} = \frac{\vec{E}_0 \times \vec{B}_{\phi 0}}{B_{\phi 0}^2} \quad .$$

The ExB velocity acts as a mode convection velocity. The drift wave phase velocity is then:

$$\vec{u}_\phi = \vec{u}_{ne} + \vec{u}_{ExB0}$$

2.3 Conditions for mode growth

3.a Graphic Description



With the previous basic model, it is not possible to conclude on the stability of the drift wave: the model only describes the mode propagation.

In order to describe the mode stability characteristic, it is necessary to take into account the ion inertia: because of their inertia, the ions have a drift velocity along the gradient direction $\vec{e}_x$ which is not exactly the ExB drift velocity described above. The velocity difference between ions and electrons then creates a slight polarization in this velocity difference direction. The plasma potential φ is modified. It is no longer in phase, but lagging behind the density n_{e1} . Because of this delay, the oscillating ExB drift velocity along $\vec{e}_x$, pushes the iso-density bumps and dips further aside: the wave is amplified. The mode propagation velocity is also modified: it is slower than the electron diamagnetic velocity.

3.b Effect of the phase shift between the potential and the density on the mode in the fluid model

We consider the equations between the plasma potential φ_1 and the density fluctuation n_{e1} from the above fluid propagation model.

In equation (2.1) , the hypothesis that potential and density are in phase :

$$\varphi_1=\frac{k_B T_e}{q_e n_{e0}} n_{e1}$$

is replaced with an arbitrary phase shift $\phi_{n\varphi}$ between the potential φ_1 and the density n_{e1} :

$$\varphi_1=\frac{k_B T_e}{q_e n_{e0}} e^{i\phi_{n\varphi}} n_{e1}$$

Since the wave time phase decreases with time ($e^{-i\omega t}$), the delay of the potential relative to the density corresponds to a positive phase shift : $0<\phi_{n\varphi}<\pi$.

The second equation between φ_1 and n_{e1} , equation (2.2), related to $\vec{E}_1\times\vec{B}_0$ drift, is unchanged:

$$\omega n_{e1}=-k_y \frac{\varphi_1}{B_{\phi 0}} \partial_x n_{e0}$$

The 2 equation combination modifies the dispersion relation. The mode angular frequency becomes complex :

$$\omega=\omega_R+i\gamma=-k_y \frac{k_B T_e}{q_e B_{\phi 0}} \frac{\partial_x n_{e0}}{n_{e0}} e^{i\phi_{n\varphi}}$$

The wave phase velocity is smaller than the electron diamagnetic velocity:

$$u_\phi=\frac{\omega_R}{k_y}=u_{ne}\cos\phi_{n\varphi} \quad .$$

In the presence of a non-zero mean electric field $\vec{E}_0$, the velocity is modified by the E cross B drift :

$$\vec{u}_\phi=\vec{u}_{ne}\cos\phi_{n\varphi}+\vec{u}_{ExB0}$$

The growth rate γ is :

$$\gamma=k_y u_{ne}\sin\phi_{n\varphi}$$

If the phase shift $\phi_{n\varphi}$ is positive, corresponding to the potential delay relative to the density, the growth rate γ has the same sign as ω_R : the mode grows in the propagation direction.

The larger the phase shift $\phi_{n\varphi}$ is (below $\pi/2$), the larger the growth rate γ is and the smaller the mode phase velocity v_ϕ is compared to the electron diamagnetic velocity u_{ne}.

This basic model only describes the effect of the phase shift between the potential and the density on the mode propagation and the mode growth.

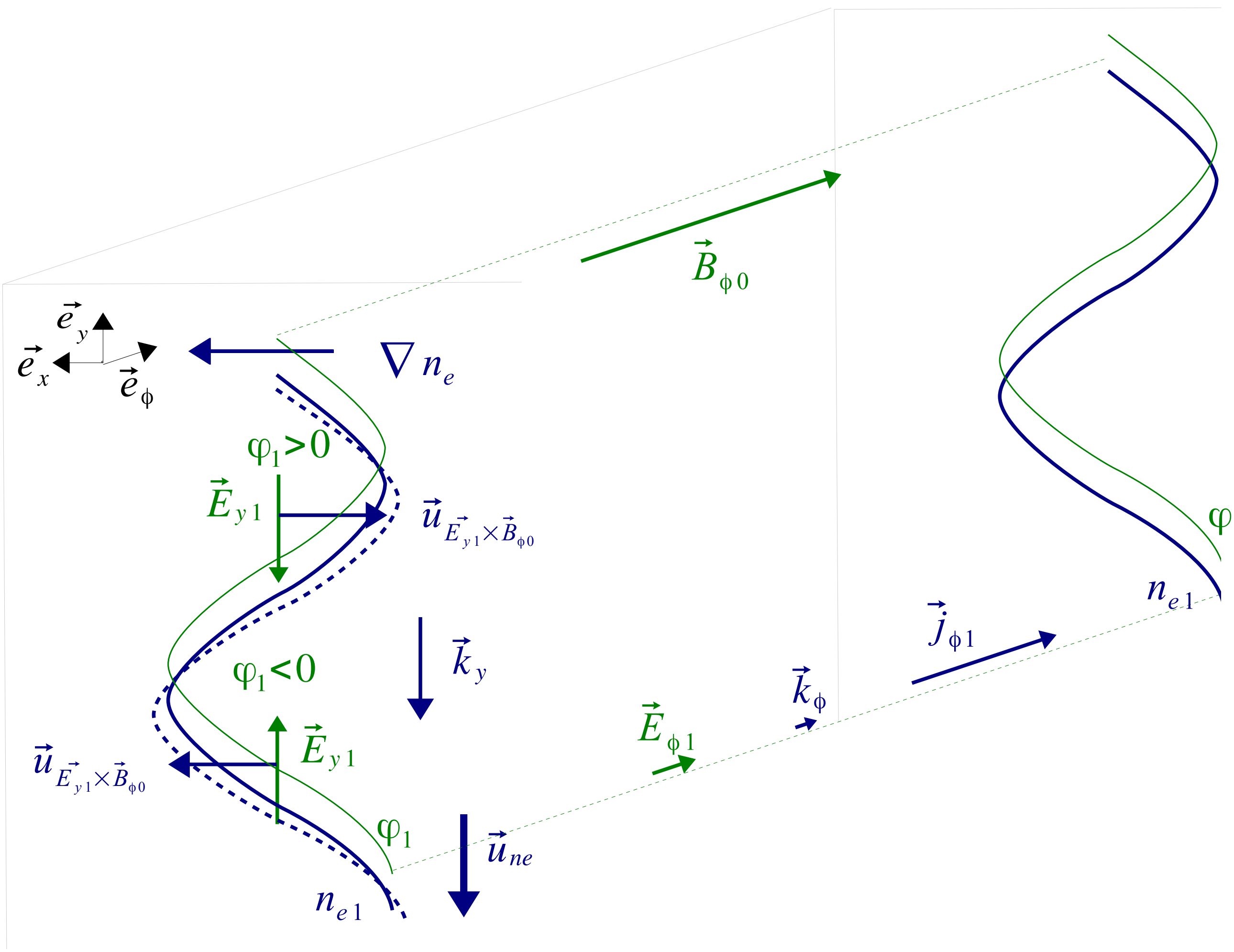
But this model does not give the physical origin of the phase shift. A further description is needed.

3 Drift Wave Linear Fluid model with dissipation

The phase shift between the plasma potential and the electron density appears when the collisions between the ions and the electrons along the magnetic field lines are taken into account. These collisions generate a delay between the electron density and the potential. A linear fluid model including the dissipation mechanism provides an estimate of the wave growth rate.

1.a The parallel current role

The fluid model is simplified: we consider a single species fluid model.



Because of the quasi-neutrality, the density is common to the ions and the electrons on the instability scale:

$$n = n_i = n_e$$

The fluid describes the ion dynamics:

$$\vec{u} = \vec{u}_i$$

This fluid velocity is in the first approximation the ExB drift velocity, common to the ions and the electrons:

$$\vec{u}_i \sim \frac{\vec{E}_1 \wedge \vec{B}_{\phi 0}}{B_{\phi 0}} \sim \vec{u}_e$$

Nevertheless at the second order, a difference can appear between the 2 species velocities. This difference generates an electric current $\vec{j}$:

$$\vec{j} = q_e n(\vec{u}_i - \vec{u}_e)$$

The conditions are the same as before. The uniform magnetic field $\vec{B}_0$ is along $\vec{e}_\phi$: $\vec{B}_{\phi 0}$. The density gradient $\vec{\nabla} n_0$ is along $\vec{e}_x$.

The fluctuation is perpendicular to the gradient. Part of it is parallel to the magnetic field :

$$n_1(\vec{r}, t) = \tilde{n}_1 e^{j(k_y y + k_\phi z_\phi - \omega t)}$$

The dissipation mechanism is a resistivity phenomenon. The resistivity η is proportional to the collision frequency of electrons on ions ν_{ei} :

$$\eta = \frac{m_e \nu_{ei}}{n_0 q_e^2}$$

We apply the generalized Ohm's law, on the ion cyclotron motion scale : the mode angular frequency ω is small compared to the ion cyclotron frequency $\omega \ll \omega_{ci}$.

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} + \frac{1}{n_0 q_e}(\vec{j} \times \vec{B} - \vec{\nabla} P_e)$$

Projected on the direction parallel to $\vec{B}$, the generalized Ohm's law is written :

$$E_{\phi 1} = \eta j_{\phi 1} - i k_\phi \frac{k_B T_e}{q_e} \frac{n_1}{n_0} \tag{3.1}$$

The electric field is no longer directly proportional to density fluctuations. The ohmic part may induce a delay between plasma potential and the plasma density.

In the perpendicular direction, on the scale of ion motion, the fluid velocity is due to the drift effect $\vec{E} \times \vec{B}$:

$$\vec{u}_\times = \frac{\vec{E} \times \vec{B}_{\phi 0}}{B_{\phi 0}^2}$$

The electric field arises from a potential $\vec{E}_1 = -\vec{\nabla} \varphi_1$. The projected velocity along the direction of the gradient, $\vec{e}_x$ is :

$$u_{x1} = \frac{E_{y1}}{B_{\phi 0}} = \frac{-i k_y \varphi}{B_{\phi 0}}$$

Along the parallel direction, $\vec{e}_z$, E_{z1} also arises from the electrostatic potential:

$$E_{\phi 1} = -i k_\phi \varphi = \frac{k_\phi}{k_y} B_{\phi 0} u_{x1} \tag{3.2}$$

We combine the two equations (3.1) and (3.2) to eliminate $E_{\phi 1}$:

$$k_z B_{\phi 0} u_{x1} = k_y \left(\eta j_{\phi 1} - i k_\phi \frac{k_B T_e}{q_e} \frac{n_1}{n_0} \right) \tag{3.3}$$

The fluid (ions + electrons) momentum equation is written:

$$n_0 m_i \partial_t \vec{u} = -k_B T_e \vec{\nabla} n_1$$

Applied along the direction parallel to $\vec{B}_{\phi 0}$:

$$-i\omega n_0 m_i u_{\phi 1} = -i k_\phi k_B T_e n_1 \quad (3.4)$$

The ion mass conservation is written :

$$\partial_t n + \vec{\nabla}.(n\vec{u}_i) = 0$$

Because of the gradient and fluctuation directions, the 1st order terms are :

$$-i\omega n_1 + \partial_x n_0\, u_{x1} + i k_\phi n_0 u_{z1} = 0 \quad (3.5)$$

By incorporating the equation (3.4) linking $u_{\phi 1}$ to n_1 in this equation (2.11), we obtain a relation between n_1 and u_{x1} :

$$\left(i\omega - \frac{i k_\phi^2 k_B T_e}{\omega m_i}\right) n_1 = \partial_x n_0 u_{x1} \quad (3.6)$$

We use this relation to simplify the equation (3.3), in order to obtain a relation between u_{x1} and $j_{\phi 1}$:

$$\left(1 - \frac{\omega k_y u_{ne}}{\omega^2 - k_\phi^2 c_s^2}\right) u_{x1} = \frac{k_y \eta}{k_\phi B_{\phi 0}} j_{\phi 1} \quad (3.7)$$

This relation introduces the ion acoustic velocity :

$$c_s = \sqrt{\frac{k_B T_e}{m_i}}$$

and the electron diamagnetic velocity :

$$u_{ne} = -\frac{k_B T_e}{q_e B_{\phi 0} l_n}$$

The electron diamagnetic velocity can also be expressed using the ion acoustic velocity:

$$u_{ne} = \frac{c_s \rho_s}{l_n}$$

This velocity expression involves ρ_s , the ion acoustic Larmor radius:

$$\rho_s = \frac{\sqrt{m_i k_B T_e}}{q_e B_{\phi 0}} = \sqrt{\frac{T_e}{3 T_i}} \rho_{ci} = \frac{c_s}{\omega_{ci}}$$

Since the ion acoustic Larmor radius is usually much smaller than the density gradient length, $\rho_s \ll l_n$, the electron diamagnetic velocity is much smaller than the ion acoustic velocity :

$$u_{ne} \ll c_s$$

Another relation between u_{x1} and $j_{\phi 1}$ is needed. We consider the fluid momentum equation (ions + electrons) :

$$n_0 m_i \partial_t \vec{u} = \vec{j}_1 \times \vec{B}_{\phi 0} - \vec{\nabla} P_1$$

This relation is applied to both the directions perpendicular to the magnetic field :

$$-i\omega n_0 m_i u_{x1} = -i k_x P_1 + B_{\phi 0} j_{y1}$$

$$-i\omega n_0 m_i u_{y1} = -i k_y P_1 - B_{\phi 0} j_{x1}$$

We combine both the equations, multiplying the second by $i k_x$ and subtracting the first, multiplied by $i k_y$. This subtraction eliminates the pressure terms :

$$-i\omega m_i (i k_x n_0 u_{y1} - i k_y n_0 u_{x1}) = -B_{\phi 0} (i k_x j_{x1} + i k_y j_{y1}) \quad (3.8)$$

The sum of the electric charge continuity equations, because of the quasi-neutrality, induces a condition on the current:

$$\vec{\nabla}.\vec{j}_1=0$$

or :

$$\vec{\nabla}.\vec{j}_1=i k_x j_{x1}+i k_y j_{y1}+i k_\phi j_{\phi 1}=0$$

This relation simplifies the right-hand side of the equation (3.8).

$$-i\omega m_i(i k_x n_0 u_{y1}-i k_y n_0 u_{x1})=i k_\phi B_{\phi 0} j_{\phi 1}$$

Moreover, the magnetic moment is an motion invariant:

$$\mu=\frac{1}{2}\frac{m_e u_\perp^2}{B_{\phi 0}}=cste$$

As the magnetic field is constant, the perpendicular velocity is incompressible :

$$i k_x u_{x1}+i k_y u_{y1}=0$$

This relation between u_{x1} and u_{y1} is used to simplify the left-hand side of the equation (3.8). After multiplying the 2 members by $-i k_y$, this equation is written :

$$i\omega n_0 m_i(k_x^2+k_y^2)u_{x1}=B_{\phi 0} k_y k_\phi j_{\phi 1} \quad (3.9)$$

From this second relation of proportionality between u_{x1} and $j_{\phi 1}$ combined with the equation (3.7) we deduce the dispersion relation :

$$\omega-k_y u_{ne}-\frac{k_\phi^2 c_s^2}{\omega}=i\frac{\eta(k_x^2+k_y^2)}{\mu_0}\frac{\omega^2-k_\phi^2 c_s^2}{k_\phi^2 u_A^2} \quad (3.10)$$

This expression introduces the Alfven velocity :

$$u_A=\frac{B_{\phi 0}}{\sqrt{\mu_0 n_0 m_i}}$$

For our typical plasmas (magnetic field of the order of $0{,}1\,T$, electron temperature of the order of $1\,eV$) :

$$u_A\gg c_s \quad .$$

In our case, the mode is perpendicular to the gradient : $k_x=0$.

Assuming that the parallel wave number of the mode is negligible :

$$k_y u_{ne}\gg k_\phi c_s$$

The phase velocity remains close to the electron diamagnetic velocity :

$$\omega_R\sim k_y u_{ne}$$

The growth rate is always positive :

$$\gamma\sim\frac{\eta}{\mu_0}\frac{k_y^4 u_{ne}^2}{k_\phi^2 u_A^2}$$

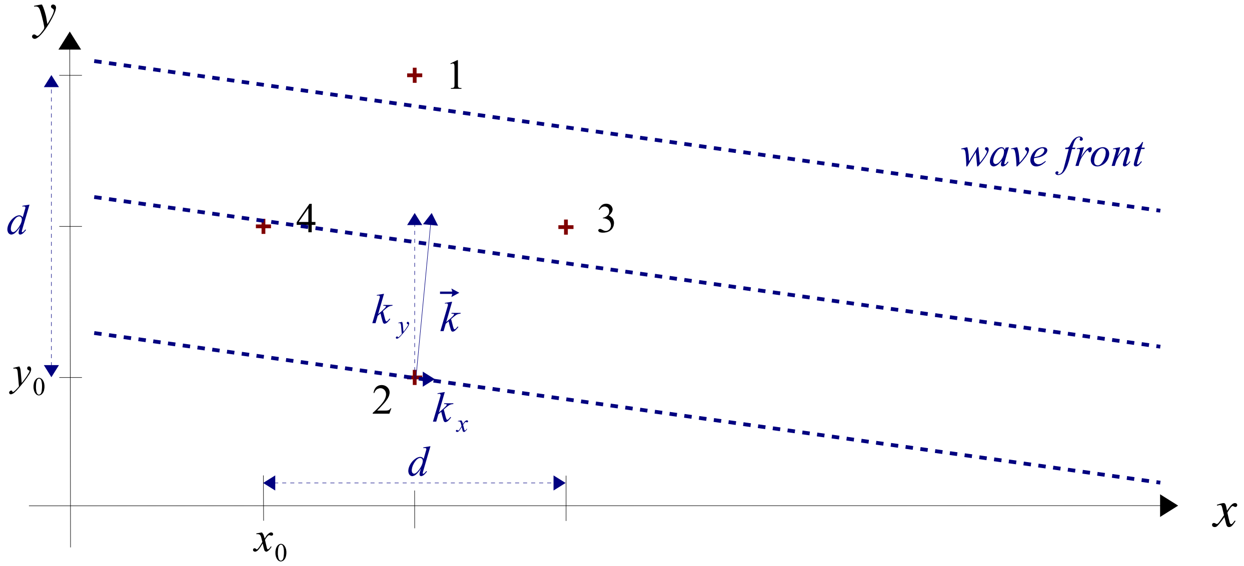
In this model, the drift mode is unstable, provided there is a sufficiently small component parallel to the magnetic field.

This model shows, physical effects along the parallel direction implies the drift mode instability. The growth rate increases indefinitely with no limit with the parallel and perpendicular wave numbers : this model is not able to give the frequency range limits for this instability.

4 Annexes

4.1 Fluctuation spatial characterization

Probe punctual time measurements give the time properties of the plasma fluctuations. In order to determine the spatial properties, several punctual measurements are necessary.



The cross-correlation or the cross-spectrum between the signals of 2 spatially spaced probes give these properties for the direction between them.

In order to characterize 2D fluctuation mode, we consider only one Fourier mode. Its angular frequency is $\omega_0 = 2\pi f_0$ and its wave vector is $\vec{k} = k_x \vec{e}_x + k_y \vec{e}_y$:

$$n_e(x,y,t) = \tilde{n}_e \cos(k_x x + k_y y - \omega_0 t) \ .$$

The phase velocity along $\vec{k}$ is $v_\phi = \omega_0 / k$.

Vertical propagation

2 probes are vertically separated by the distance d . Probe 2 is located at a position (x_0, y_0) . Probe 1 , higher, is at a position $(x_0, y_0 + d)$.

The time signal on probe 2, $n_{e2}(t) = n_e(x_0, y_0, t)$, is :

$$n_{e2}(t) = \tilde{n}_e \cos(k_x x_0 + k_y y_0 - \omega_0 t) \ .$$

The signal on probe 1 is spatially shifted, $n_{e1}(t) = n_e(x_0, y_0 + d, t)$. so:

$$n_{e1}(t) = \tilde{n}_e \cos(k_x x_0 + k_y y_0 - \omega_0 t + k_y d) \ .$$

This spatial shift between probes induces a phase shift between both signals:

$$\Delta\phi(\omega) = k_y d = \frac{d\,\omega_0}{v_{y\phi}}$$

This can also be regarded as a time delay between both signals :

$$n_{e1}(t) = n_{e2}\left(t - \frac{k_y d}{\omega_0}\right) \ .$$

The signal from probe 1 lags behind probe 2 by:

$$\tau_z = \frac{k_y d}{\omega_0} = \frac{d}{v_{y\phi}} \quad .$$

This delay can be estimated by seeking the maximum of the time cross-correlation between probe signals 1 and 2 . This method assumes the phase velocity is the same for all frequencies present in the common signal.

The phase shift can be analyzed more finely, frequency by frequency, by the cross-spectrum between the signals. Each signal Fourier transform includes a phase shift :

$$n_{e2}(\omega) \propto \delta(\omega - \omega_0) \tilde{n}_e e^{j(k_x x_0 + k_y y_0)} \quad ,$$

$$n_{e1}(\omega) \propto \delta(\omega - \omega_0) \tilde{n}_e e^{j(k_x x_0 + k_y y_0 + k_y d)} \quad .$$

The cross-spectrum between probe signals 1 and 2 is defined by:

$$S_{12}(\omega) \propto n_{e2}(\omega)^* n_{e1}(\omega)$$

$$S_{12}(\omega) \propto \delta(\omega - \omega_0) \tilde{n}_e^2 e^{j k_y d} \quad .$$

The cross-spectrum phase corresponds to the phase shift between the signals for each frequency :

$$\Delta \phi_y(\omega_0) = k_y(\omega_0) d \quad .$$

For each frequency, we can deduce k_y , the vertical wavenumber :

$$k_y(\omega_0) = \frac{\Delta \phi_y(\omega_0)}{d} \quad ,$$

The phase velocity depends on the wave vector component in the vertical direction:

$$v_{y\phi}(\omega_0) = \frac{\omega_0}{k_y(\omega_0)} \quad .$$

$v_{y\phi}(\omega_0)$ might be specific to each frequency:

$$v_{y\phi}(\omega_0) = \frac{d\,\omega_0}{\Delta \phi_y(\omega_0)} \quad .$$

If modes are present for a large range of frequencies, and if the phase velocity is independent of the frequency, there is a linear relationship between the cross spectrum phase and the frequency with the slope $\frac{d v_{y\phi}}{d\omega} = \frac{d}{v_{y\phi}}$.

The radial component

The 1D phase velocity is: $v_{x\phi}(\omega_0) = \frac{\omega_0}{k_x(\omega_0)}$.

The delay between radially separated probe 3 and 4 (distance d) is:

$$\tau_x = \frac{k_x d}{\omega_0} = \frac{d}{v_{x\phi}} \quad .$$

The phase of the cross-spectrum between probe signal 3 and 4 is:

$$\Delta \phi_x(\omega_0) = k_x(\omega_0) d = \frac{d}{v_{x\phi}(\omega_0)} \omega_0$$

The radial wavenumber and phase velocity are:

$$k_x(\omega_0) = \frac{\Delta \phi_x(\omega_0)}{d} \quad ,$$

$$v_{x\phi}(\omega_0) = \frac{d\,\omega_0}{\Delta \phi_x(\omega_0)} \quad .$$

Since the phase velocity is inversely proportional to the wavenumber, and $\vec{k}=k_x\vec{e}_x+k_y\vec{e}_y$, the relation between the phase velocity component is:

$\frac{1}{v_\phi}\vec{e}_k=\frac{1}{v_{y\phi}}\vec{e}_y+\frac{1}{v_{x\phi}}\vec{e}_x$ (where $\vec{e}_k=\frac{1}{k}\vec{k}$).

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Physical constants

$k_B = 1,38\ 10^{-23} JK^{-1}$: Boltzmann constant

$h = 6.62\ 10^{-34} Js$: Planck constant

$C = 2,99\ 10^{8} ms^{-1}$: speed of light in vacuum

$\varepsilon_0 = 8.85\ 10^{-12} Fm^{-1}$: vacuum permittivity

$\mu_0 = 4\pi\ 10^{-7} Hm^{-1}$: vacuum permeability

$q_e = 1,60\ 10^{-19} C$: elementary charge

$m_e = 9,11\ 10^{-31} kg$: electron mass

$r_e = \frac{1}{4\pi\epsilon_0}\frac{q_e^2}{m_e C^2} = 2.82\ 10^{-15} m$: electron classical radius

$N_A = 6,022\ 10^{23} mol^{-1}$: Avogadro constant

$m_u = 1,66\ 10^{-27} kg$: atomic mass unit

$m_i = 40\, m_u = 66,4\ 10^{-27} kg$: Argon atomic mass Ar_{40}^{18}

Standard parameters

$T_0 = 273.15 K$: standard air temperature ($0\,°C$)

$P_0 = 1,013\ 10^5 Pa$: standard air pressure

$n_0 = 2,69\ 10^{25} m^{-3}$: ideal gas molecular density at T_0 and P_0

Units

$1\, Torr = \frac{1,013\ 10^5}{760} Pa = 133,3\, Pa$: pressure corresponding 1 mm of mercury

$1\, eV = \frac{1,6\ 10^{-19}}{1,38\ 10^{-23}} K = 1,16\ 10^4 K$