INTRODUCTION TO
RANDOM SIGNALS

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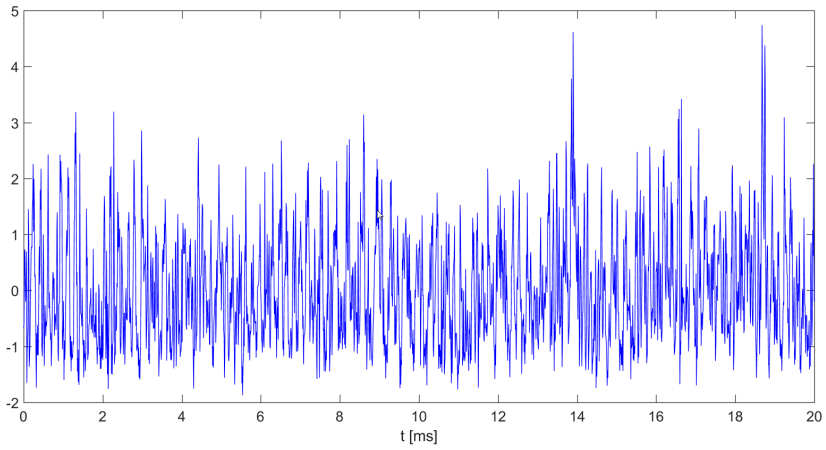

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Introduction

Random time signal properties are introduced.

In the first part we present the difference between finite energy and finite power signals, and their frequency spectral and time correlation properties.

The second part shows the effect of the signal time sampling and the digital filtering.

The last part is about random variable probability distribution properties (with no time variation): the characteristic function, the moments and the cumulants.

Contents

2 Finite power signal

2.1 Finite energy or finite power signals

Finite energy signal

The finite energy signal $u(t)$ corresponds to the usual case, where the energy (the signal variance) integrated over the whole time converges :

$$\int_{-\infty}^{+\infty} |u(t)|^2 dt < +\infty$$

We can apply the classical Fourier analysis. We introduce it in the section 2.2.

Finite power signal

The Fourier analysis is not directly possible, in the case where the signal energy diverges. This is typically the case for time signals from instruments observing unstable or turbulent continuous media.

In this case, the signal $s(t)$ variance diverges: $\int_{-\infty}^{+\infty} |s(t)|^2 dt = +\infty$

Nevertheless, we can adapt the Fourier analysis if the time mean signal variance has a finite value when the integration time tends towards infinity,

$$\lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt < +\infty$$

In this case, the signal is a finite power signal.

It is possible to define the power spectrum, an adaptation of the energy spectrum. The definition of correlation can also be adapted to finite power signals.

2.2 Finite energy signal spectral properties

2.a Fourier transform, autocorrelation and spectrum

For a finite energy signal, the signal Fourier transform is defined:

$$\hat{u}(\omega) = \int_{-\infty}^{+\infty} u(t) e^{-i\omega t} dt$$

Its expression in the frequency domain differs :

$$\check{u}(f) = \int_{-\infty}^{+\infty} u(t) e^{-i 2\pi f t} dt = \hat{u}(2\pi f)$$

The signal energy autospectrum is the Fourier transform squared modulus:

$$U(\omega) = |\hat{u}(\omega)|^2$$

The signal time autocorrelation is:

$$C(\tau) = \int_{-\infty}^{\infty} u^*(t) u(t+\tau) dt$$

There is a direct relation by Fourier transform between the autocorrelation and the autospectrum:

$$S(f) = \int_{-\infty}^{+\infty} d\tau C(\tau) e^{-i 2\pi f \tau}$$

The relation between both is verified by change of variables inside the integrals. The calculation is equivalent to that for finite power signals that is showed in section 3.b.

Parseval's relation is applied to finite energy signals :

$$\int_{-\infty}^{+\infty} |u(t)|^2 dt = C(0) = \int_{-\infty}^{+\infty} U(f)\, df$$

2.b Uncertainty relation

An important property of the Fourier transform is the uncertainty relation. It is deduced from the Cauchy-Schwarz inequality [Appel2007]. It is applicable if the integrals $\int_{-\infty}^{+\infty} t^2 |u(t)|^2 dt$ and $\int_{-\infty}^{+\infty} |u'(t)|^2 dt$ exist.

In this case, the time domain variances are :

$$\langle t^2 \rangle = \frac{1}{\int_{-\infty}^{+\infty} |u(t)|^2 dt} \int_{-\infty}^{+\infty} t^2 |u(t)|^2 dt$$

and the frequency domain ones :

$$\langle \omega^2 \rangle = \frac{1}{\int_{-\infty}^{+\infty} |\hat{u}(\omega)|^2 d\omega} \int_{-\infty}^{+\infty} \omega^2 |\hat{u}(\omega)|^2 d\omega = \frac{16\pi^4}{\int_{-\infty}^{+\infty} |u(t)|^2 dt} \int_{-\infty}^{+\infty} |u'(t)|^2 dt$$

they verify the relation :

$$\boxed{\langle t^2 \rangle \langle \omega^2 \rangle \geq \frac{1}{4}}$$

Furthermore $\langle t^2 \rangle \langle \omega^2 \rangle = 1/4$ if and only if $u(t)$ is a centered Gaussian function.

$\langle t^2 \rangle$ and $\langle \omega^2 \rangle$ are calculated from the integral of the squared functions : $|u(t)|^2$ and $|\hat{u}(\omega)|^2$.

For a centered Gaussian function of variance σ^2 :

$$u(t) = e^{-t^2/2\sigma^2}$$

the Fourier transform is also Gaussian :

$$\hat{u}(\omega) = e^{-\omega^2\sigma^2/2}$$

The 2 widths are half of the variances of the Gaussian :

$$\langle t^2 \rangle = \frac{\sigma^2}{2} \text{ and } \langle \omega^2 \rangle = \frac{1}{2\sigma^2}$$

The uncertainty relation is in the Gaussian case an equality.

Consequently for the autocorrelation and the autospectrum, the narrower is the time width of the autocorrelation, the wider is the frequency width of the autospectrum.

2.3 Spectral properties of finite power signals

3.a Finite power signal spectral density

The time signals from instruments for observing unstable or even turbulent media are stationary random time signals: over time longer than the signal autocorrelation time, assumed to be finite, the signal finite time mean variance is stationary.

The energy of such a signal $s(t)$ diverge : $\int_{-\infty}^{+\infty} |s(t)|^2 dt = +\infty$

It is not possible to calculate the energy spectrum using the Fourier transform for infinite time.

On the other hand, their power, the time mean squared value over a finite duration, is finite. We can define the power of the signal by :

$$\langle |s(t)|^2 \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt \tag{2.1}$$

In order to analyze the frequencies of the signal, we can perform a Fourier transform over a finite time T :

$$\hat{s_T}(\omega) = \int_{-T/2}^{T/2} s(t) e^{-i\omega t} dt$$

We also use the expression on the frequency:

$$\check{s_T}(f) = \int_T s(t) e^{-i2\pi f t} dt = \hat{s_T}(2\pi f)$$

We introduce the signal frequency power spectrum:

$$S(\omega) = \lim_{T \to +\infty} \frac{1}{T} |\hat{s_T}(\omega)|^2 \tag{2.2}$$

By its quadratic form, unlike the Fourier transform, the properties of the spectral density are only related to the stationary parameters of the signal. They are independent for example of the choice of the time origin for the Fourier transform.

Spectrum and power time normalization

Spectrum normalization is $1/T$ and not $1/T^2$. For 2 distinct time intervals $I_1 = [T ; T']$ and $I_2 = [T' ; T'']$ larger than the signal correlation time, the squared modulus of $s(\omega)$ on the union of both time intervals is :

$$|\hat{s_{1+2}}(\omega)|^2 = (\hat{s_1}(\omega) + \hat{s_2}(\omega))(\hat{s_1^*}(\omega) + \hat{s_2^*}(\omega))$$

after multiplication expansion :

$$|\hat{s_{1+2}}(\omega)|^2 = |\hat{s_1}(\omega)|^2 + |\hat{s_2}(\omega)|^2 + 2\Re(\hat{s_1}(\omega)\hat{s_2^*}(\omega))$$

For uncorrelated intervals, $\hat{s_1}(\omega)\hat{s_2^*}(\omega) = 0$:

$$|\hat{s_{1+2}}(\omega)|^2 = |\hat{s_1}(\omega)|^2 + |\hat{s_2}(\omega)|^2$$

The squared modules of $s_T(\omega)$ on uncorrelated intervals, are added. On time T , much larger than the signal correlation time, the modulus squared $|\hat{s_T}(\omega)|^2$ does not grow as T^2 , but increases linearly with time T .

3.b Time autocorrelation and frequency spectrum

The signal $s(t)$ time autocorrelation is :

$$C(\tau)=\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}s^*(t)\,s(t+\tau)\,dt \qquad (2.3)$$

Relation between autocorrelation and spectral density

For finite power signals, autocorrelation and power spectrum are related by Fourier transform (Wiener-Khintchine relation) .The power spectrum is:

$$S(f)=\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}s^*(t)\,e^{i2\pi ft}\,dt\ \int_{-T/2}^{T/2}s(t')\,e^{-i2\pi ft'}\,dt'$$

2 integrations are nested :

$$S(f)=\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}s^*(t)\,s(t')\,e^{-i2\pi f(t'-t)}\,dt\,dt'$$

A change of variable is necessary : $t'\to\tau \quad \tau=t'-t \quad d\tau=dt'$:

$$S(f)=\lim_{T\to+\infty}\frac{1}{T}\int_{-T}^{T}\int_{min(-T/2,\tau)}^{max(T/2+\tau,T/2)}s^*(t)\,s(t+\tau)\,e^{-i2\pi f\tau}\,dt\,d\tau$$

Factors are rearranged :

$$S(f)=\lim_{T\to+\infty}\int_{-T}^{T}\left(\frac{1}{T}\int_{min(-T/2,\tau)}^{max(T/2+\tau,T/2)}s^*(t)\,s(t+\tau)\,dt\right)e^{-i2\pi f\tau}\,d\tau$$

Hence the relation between the frequency spectrum and time autocorrelation :

$$S(f)=\int_{-\infty}^{+\infty}d\tau\,C(\tau)\,e^{-i2\pi f\tau} \qquad (2.4)$$

Parseval's formula

The equivalent of Parseval's formula for finite energy signals relates the signal time mean squared absolute value to the integral of the spectrum :

$$\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|s(t)|^2\,dt=C(0)=\int_{-\infty}^{+\infty}S(f)\,df \qquad (2.5)$$

Uncertainty relationship

There is no direct equivalent to the uncertainty relation for finite power signals. Nevertheless, as the spectrum is the Fourier transform of the autocorrelation, the uncertainty relation could be applied to the autocorrelation: it then gives a relation between the width of the autocorrelation and the width of the power spectrum.

3.c Cross-spectrum and cross-correlation

We consider 2 time real finite power signals $s_a(t)$ and $s_b(t)$.

The frequency cross-spectrum between both signals is:

$$S_{ab}(f)=\lim_{T\to+\infty}\frac{1}{T}\hat{s_{aT}}(f)^*\,\hat{s_{bT}}(f)$$

Cross-spectrum phase

The cross-spectrum includes for each frequency the phase shift between the signals contrary to the signal autospectrum.

2 complex signals, simplified to a single frequency mode, phase shifted by $\varphi_2-\varphi_1$:

$$s_a(t)=\widetilde{s}_a e^{j(2\pi f_0 t+\varphi_1)}$$

and

$$s_b(t)=\widetilde{s}_b e^{j(2\pi f_0 t+\varphi_2)}$$

This phase shift is present in the argument of the cross-spectrum :

$$Arg\left(S_{ab}(f_0)\right)=\varphi_2-\varphi_1$$

If the cross-spectrum argument positive for a positive frequency $f_0>0$, this phase shift is positive, $\varphi_2-\varphi_1>0$, the signal $s_b(t)$ is ahead of $s_a(t)$.

3.d Cross-correlations between 2 signals

The time cross-correlation function between 2 real (or complex) signals, depending on the delay between signals, allows a temporal study of the delay between signals:

$$C_{ab}(\tau)=\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} s_a^*(t)s_b(t+\tau)\,dt$$

Cross-spectrum and cross-correlation are linked by the Fourier transform:

$$S_{ab}(f)=\int_{-\infty}^{+\infty} C_{ab}(\tau)\,\mathrm{e}^{-i2\pi f\tau}\,d\tau$$

Correlation between normalized signals

The signal is normalized by subtracting its mean value and dividing it by its standard deviation :

$$\breve{s}_a(t)=\frac{s_a(t)-\langle s_a\rangle}{\sigma_a}$$

For normalized signals, the signal autocorrelation at $\tau=0$ is :

$$C_{\breve{a}\breve{a}}(0)=1$$

For any τ , the cross-correlation between 2 normalized real signals, verifies:

$$-1\le C_{\breve{a}\breve{b}}(\tau)\le 1$$

This is a analogous to the Cauchy-Schwarz inequality.

Cross-correlation rate and delay between signals

The more the signals resemble each other, the larger the cross-correlation is. We reduce it to a single value, corresponding to the maximum of the absolute value of the cross-correlation:

$$C_{\breve{a}\breve{b},max}=max_\tau\left|C_{\breve{a}\breve{b}}(\tau)\right|$$

the delay τ_{max} for which this maximum is reached provides information on the delay between both signals, if the signals are correlated with each other. If this time is positive $\tau_{max}>0$, the signal $s_b(t)$ is late on $s_a(t)$.

Uncorrelated signals

If the normalized signals cross-correlation function is close to zero at any time the signals are uncorrelated.

This does not mean that the signals are statistically independent.

For 2 signals whose frequency ranges are disjoint, the cross-spectrum is in fact

zero. By Fourier transform, the cross-correlation is also zero. The 2 signals can nevertheless be dependent, for example in the case where the envelope of the signal at a faster frequency is correlated to the second signal at a slower frequency.

Even for 2 signals at the same frequency. The cross-correlation is zero when both signals are 90° out of phase (in quadrature).

3 Digital discrete-time signals

3.1 Digitizing a signal

The development of computing led to the development of digital discrete-time signal processing. An essential step becomes the digitization of the real time signal. It is characterized by 2 complementary effects.

Time sampling: The time signal is no longer a continuous-time signal, but a time series.

Signal value digitization: The signal can no longer take any real value, but only a finite number of values, regularly spaced.

1.a Sampling

The initial signal is a continuous-time signal over time $s(t)$. Sampling consists of extracting a regular time series x_n . The time step is the sampling period t_s :

$$x_n = s(n t_s)$$

The operation is equivalent to multiplying the signal by a Dirac comb :

$$\Psi_{t_s}(t) = \sum_{n=-\infty}^{n=+\infty} \delta(t - n t_s)$$

The sampled signal is:

$$\boxed{x(t) = s(t) t_s \sum_{n=-\infty}^{n=+\infty} \delta(t - n t_s)} \quad . \tag{3.1}$$

Multiplication by t_s conserves the energy (or power) of the sampled signal.

But in this case, the discrete-time signal has a different unit than the continuous-time one.

Sampling frequency properties

The Fourier transform of the Dirac comb in time is a Dirac comb in frequency:

$$\check{\Psi}_{t_s}(f) = f_s \sum_{m=-\infty}^{m=+\infty} \delta(f - m f_s)$$

where f_s is the sampling frequency :

$$f_s = \frac{1}{t_s}$$

The Fourier transform of the sampled signal is the convolution product of the Fourier transform with a Dirac comb :

$$\check{x}(f) = \check{s}(f) * \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

or :

$$\check{x}(f)=\sum_{m=-\infty}^{+\infty}\check{s}(f-m f_s) \tag{3.2}$$

This property is the frequency folding: the value of the Fourier transform at a certain frequency f is the sum of all Fourier transforms of the original continuous-time signal at frequencies shifted by an integer number of times the sampling frequency $f-m f_s$. The signal sampling makes it impossible to distinguish these frequencies.

Nyquist criterion

In order to correctly determine the sample rate f_s , it is necessary to know, if it exists, the continuous-time signal maximum frequency f_{max} :

$\check{s}(f)=0 \;\; if \; |f|>f_{max}$

In order to avoid the frequency aliasing, this maximum frequency must verify :

$$f_{max}<\frac{1}{2}f_s \;.$$

This is the Nyquist criterion.

Sampling (or Shannon's) theorem

With this condition, the frequency aliasing is avoided. The initial signal can even be deduced from the sampled signal. There is no information loss due to sampling :

$$s(t)=\sum_{n=-\infty}^{n=+\infty}x_n\frac{\sin\left(\pi(\frac{t}{t_s}-n)\right)}{\pi(\frac{t}{t_s}-n)} \tag{3.3}$$

3.2 Digital Power Spectrum and Coherence

We consider a finite power discrete-time signal a_n . The sampling frequency is $f_s=1/t_s$ and its unit is $[A]$.

We introduce finite approximations of the power spectrum and correlation for signal finite data number.

2.a Fourier Transform

The continuous-time signal Fourier transform integration formula:

$\hat{s_T}(f)=\int_{-T/2}^{T/2}s(t)\,\mathrm{e}^{-i\,2\pi f\,t}\,dt$

is replaced with the finite sum of N_{Ft} indices:

$$\check{a}_l=t_s\sum_{k=1}^{N_{Ft}}a_k\,e^{-j\frac{2\pi}{N_{Ft}}(l-1)(k-1)} \tag{3.4}$$

The discrete-time Fourier transformed signal has the unit $[AHz^{-1}]$.

Most numerical Fourier transform algorithms omit the time factor t_s in order to be completely time and frequency independent. In this case, the unit is simply $[A]$

Time and frequency ranges

The discrete-time Fourier transform exponential does not depend on time or frequency parameters.

The time and frequency units reappear if we try to extract the corresponding time and frequency series.

The signal time series corresponds to the data series:

$t_k=(k-1)t_s$ for $1\le k\le N_{Ft}$.

The time origin corresponds to the first sample. The total time is $N_{Ft}t_s$.

The Fourier transform complementary frequency series is:

$$f_l=\frac{(l-1)}{N_{Ft}}f_s \quad .$$

Because of the frequency aliasing, the Fourier transform is periodic with the period with period f_s , corresponding to the number N_{Ft} . The frequency series is restricted to $1\le l\le N_{Ft}$.

The discrete-time Fourier transform expression with time and frequency series is:

$$\breve{a}_l=t_s\sum_{k=1}^{N_{Ft}}a_k e^{-j2\pi f_l t_k} \quad .$$

Frequency resolution and range

The discrete Fourier transform frequency range is:

$$0\le f_l\le\frac{N_{Ft}-1}{N_{Ft}}f_s \quad .$$

The Fourier transform frequency resolution is $\frac{1}{N_{Ft}}f_s$. The frequency resolution is the inverse of the Fourier transformed series total time $N_{Ft}t_s$.

Because of the frequency aliasing, the frequency range can be changed to $[-\frac{1}{2}f_s;\left(\frac{1}{2}-\frac{1}{N_{Ft}}\right)f_s]$ after interverting both halves of the Fourier transformed series if N_{Ft} is even:

$$\breve{a}'=[\breve{a}_{1+N_{Ft}/2}\ldots\breve{a}_{N_{Ft}},\breve{a}_1\ldots\breve{a}_{N_{Ft}/2}]$$

with

$$f_l=\left[\frac{(l-1)}{N_{Ft}}-\frac{1}{2}\right]f_s \quad .$$

This frequency series is more appropriate with the Nyquist criterion.

Inverse Fourier transform

The continuous-time version of inverse Fourier transform is

$$s_T(t)=\int_{-\infty}^{\infty}s(f)\mathrm{e}^{i2\pi ft}df$$

Because the frequency resolution is $\frac{f_s}{N_{Ft}}$, the inverse Fourier transform of $\breve{a}_l$ is:

$$a_k=\frac{f_s}{N_{Ft}}\sum_{l=1}^{N_{Ft}}\breve{a}_l e^{j\frac{2\pi}{N_{Ft}}(k-1)(l-1)}$$

The normalized version of the discrete inverse Fourier transform algorithms omit the f_s frequency factor.

Fast Fourier Transform algorithms

The discrete-time Fourier transform calculation needs N_{Ft} multiplications for all N_{Ft} indices of the Fourier transformed series $\check{a}_l$. The multiplication number is for the most naive algorithm, N_{Ft}^2 .

When N_{Ft} can be factorized, the calculation can be done separately on regular smaller data segments. The results can be recombined with less multiplications than the naive algorithm. For the most efficient algorithm, if N_{Ft} is a power of 2, the number of multiplications is of the order of $N_{Ft}\log_2(N_{Ft})$. For $N_{Ft}=1024$, the multiplication number is reduced by a factor $\log_2(N_{Ft})/N_{Ft}\sim 1/100$.

2.b Power Auto- and Cross-Spectral Density

We estimate the frequency power spectrum using data finite sample number.

The Fourier transform data number N_{Ft} choice depends on the expected frequency resolution $\frac{1}{N_{Ft}} f_s$.

If the maximum data number is much larger than N_{Ft} , the discrete time power spectrum can be averaged over consecutive segments of N_{Ft} data. This averaging method is also known as the standard periodogram. The segment number is N_{sg} .

The finite energy or finite power condition on the signal has no meaning for the finite discrete-time signals. Nevertheless we consider signals tending to the finite power case : signal statistics like the mean value or standard deviation are comparable between segments.

Autospectrum

The discrete-time Fourier transform is applied separately on each signal segment $g=1\ldots N_{sg}$ and for each frequency $f_l=\frac{(l-1)}{N_{Ft}} f_s$:

$$\check{a}_{l,g}=t_s\sum_{k=1}^{N_{Ft}} a_{(g-1)N_{Ft}+k}\, e^{-j\frac{2\pi}{N_{Ft}}(k-1)(l-1)} .$$

The continuous-time signal power autospectrum function:

$$S(\omega)=\lim_{T\to+\infty}\frac{1}{T}\left|\hat{s}_T(\omega)\right|^2$$

is replaced with the segment average power autospectrum series.

For each segment, the formula is:

$$\check{S}_a[l,g]=\frac{1}{N_{Ft}t_s}\left|\check{a}_{l,g}\right|^2 .$$

The result is averaged on all segments :

$$\check{S}_a[l]=\frac{1}{N_{sg}}\sum_{g=1}^{N_{sg}}\check{S}_a[l,g]=\frac{t_s}{N_{sg}N_{Ft}}\sum_{g=1}^{N_{sg}}\left|\sum_{k=1}^{N_{Ft}} a_{(g-1)N_{Ft}+k}\, e^{-j\frac{2\pi}{N_{Ft}}(k-1)(l-1)}\right|^2 . \qquad (3.5)$$

The frequency series is the same as the Fourier transform one: $f_l = \frac{(l-1)}{N_{Ft}} f_s$.

The spectrum unit is $[A^2 Hz^{-1}]$.

Cross-spectrum

We consider a 2nd discrete-time signal b_n synchronized with a_n (same sampling frequency $f_s = 1/t_s$, unit $[B]$).

The signal b_n segmented discrete-time Fourier transform is (unit $[BHz^{-1}]$):

$$\check{b}_{l,g} = t_s \sum_{k=1}^{N_{Ft}} b_{(g-1)N_{Ft}+k} e^{-j\frac{2\pi}{N_{Ft}}(k-1)(l-1)}$$

The cross-spectrum is defined by replacing the signal squared modulus with the product of the 1st signal Fourier transformed conjugate with the 2nd signal Fourier transformed.

For one segment, the cross spectrum is defined as:

$$\check{S}_{ab}[l,g] = \frac{1}{N_{Ft} t_s} \check{a}^*_{l,g} \check{b}_{l,g}$$

The value for N_{sg} segments is:

$$\check{S}_{ab}[l] = \frac{1}{N_{sg} N_{Ft} t_s} \sum_{g=1}^{N_{sg}} \check{a}^*_{l,g} \check{b}_{l,g} \qquad (3.6)$$

The power cross-spectrum unit is $[AB\, Hz^{-1}]$.

2.c Correlation

Autocorrelation

Because the discrete-time Fourier transform is defined only for discrete frequency values means the time signal used for the Fourier transform is considered as periodic. For the first segment with N_{Ft} data, the Fourier transform considers the signal as $a'_k = a_{mod(k,N_{Ft})}$.

Because of this, the discrete-time signal autocorrelation is defined as circular.

The continuous-time signal autocorrelation

$$C(\tau) = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} s^*(t) s(t+\tau)\, dt$$

is replaced, for the first segment, with:

$$C_a[m,1] = \frac{1}{N_{Ft}} \sum_{k=1}^{N_{Ft}} a^*_k a_{mod(k+m-1,N_{Ft})} \quad \text{for} \quad m = 1 \ldots N_{Ft} \qquad (3.7)$$

For each segment separately, the time autocorrelation is:

$$C_a[m,g] = \frac{1}{N_{Ft}} \sum_{k=1}^{N_{Ft}} a^*_{(g-1)N_{Ft}+k} a_{(g-1)N_{Ft}+mod(k+m-1,N_{Ft})}$$

The mean value for N_{sg} segments is:

$$C_a[m] = \frac{1}{N_{sg}} \sum_{g=1}^{N_{sg}} C_a[m,g]$$

$$C_a[m]=\frac{1}{N_{sg}N_{Ft}}\sum_{g=1}^{N_{sg}}\sum_{k=1}^{N_{Ft}}a^*_{(g-1)N_{Ft}+k}a_{(g-1)N_{Ft}+mod(k+m-1,N_{Ft})}$$

The autocorrelation unit is $[A^2]$.

Correlation Time range

The corresponding time series is limited to N_{Ft} :

$\tau_m=(m-1)t_s$ for $m=1...N_{Ft}$

The delay range is $[0;(N_{Ft}-1)t_s]$ .The delay $\tau=0$ corresponds to $m=1$.

Because of the correlation time periodicity, if N_{Ft} is even, the delay range can be changed to $\left[-\frac{N_{Ft}}{2}t_s;\left(\frac{N_{Ft}}{2}-1\right)t_s\right]$ after interverting both halves of the Correlation series

$$C'_a=\left[C_a[1+N_{Ft}/2]...C_a[N_{Ft}],C_a[1]...C_a[N_{Ft}/2]\right]$$

with

$\tau_m=(m-1-N_{Ft}/2)t_s$.

This time series is more appropriate with the usual short time signal correlation.

Cross-correlation

For one segment the discrete-time signal cross-correlation is:

$$C_{ab}[m,1]=\frac{1}{N_{Ft}}\sum_{k=1}^{N_{Ft}}a^*_k b_{mod(k+m-1,N_{Ft})} \text{ for } m=1...N_{Ft}$$

For N_{sg} segments, the discrete-time cross-correlation is:

$$C_{ab}[m]=\frac{1}{N_{sg}N_{Ft}}\sum_{g=1}^{N_{sg}}\sum_{k=1}^{N_{Ft}}a^*_{(g-1)N_{Ft}+k}b_{(g-1)N_{Ft}+mod(k+m-1,N_{Ft})}$$

The cross-correlation unit is $[AB]$.

Relation between spectrum and correlation

The Fourier transform relation between the spectrum and the correlation for continuous-time signals

$$S(f)=\int_{-\infty}^{+\infty}d\tau C(\tau)e^{-i2\pi f\tau}$$

has an equivalent relation between the discrete-time signal spectrum and the correlation

$$\check{S}_a[l]=t_s\sum_{m=1}^{N_{Ft}}C_a[m]e^{-j\frac{2\pi}{N_{Ft}}(m-1)(l-1)}$$

and

$$\check{S}_{ab}[l]=t_s\sum_{m=1}^{N_{Ft}}C_{ab}[m]e^{-j\frac{2\pi}{N_{Ft}}(m-1)(l-1)} .$$

Because of the fast Fourier transform algorithm optimization, it is faster to compute the correlation series through the spectrum than directly from the time series.

Parseval equivalence

The Parseval relation for time continuous-time signal is:

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} |s(t)|^2\, dt = C(0) = \int_{-\infty}^{+\infty} S(f)\, df$$

The equivalent relation for discrete-time signals for autospectrum is

$$\frac{1}{N_{Ft}\, N_{sg}} \sum_{n=1}^{N_{sg} N_{Ft}} |a_n|^2\, dt = C_a[1] = \frac{1}{N_{Ft}} f_s \sum_{l=1}^{N_{Ft}} \check{S}_a[l]$$

and for cross-spectrum

$$\frac{1}{N_{Ft}\, N_{sg}} \sum_{n=1}^{N_{sg} N_{Ft}} |a_n||b_n|\, dt = C_{ab}[1] = \frac{1}{N_{Ft}} f_s \sum_{l=1}^{N_{Ft}} \check{S}_{ab}[l] \quad .$$

2.d Phase time Coherence

The coherence is the normalized version of discrete-time signal cross-spectrum

For 1 segment this is defined as :

$$\check{c}_{ab}[l, g] = \frac{\check{a}^*_{l,g}\, \check{b}_{l,g}}{|\check{a}^*_{l,g}||\check{b}_{l,g}|}$$

For 1 segment, the coherence modulus is one. The coherence phase is, for each discrete frequency, the phase shift between both signals:

$$\check{c}_{ab}[l, g] = e^{j\varphi_{l,g}}$$

For N_{sg} segments, the expression is

$$\check{c}_{ab}[l] = \frac{1}{N_{sg}} \sum_{g=1}^{N_{sg}} \frac{\check{a}^*_{l,g}\, \check{b}_{l,g}}{|\check{a}^*_{l,g}||\check{b}_{l,g}|}$$

or

$$\check{c}_{ab}[l] = \frac{1}{N_{sg}} \sum_{g=1}^{N_{sg}} e^{j\varphi_{l,g}}$$

If the phase is the same for each segment, $e^{j\varphi_{l,g}} = e^{j\varphi_l}$,the coherence modulus is:

$$|\check{c}_{ab}[l]| = 1$$

If the phase is completely random for each segments, the sum of the phases on each segment is like a random walk with normed steps and completely random directions on the complex plane. The coherence modulus, for a (large) segment number N_{sg} ,is of the order of :

$$|\check{c}_{ab}[l]| \sim \frac{1}{\sqrt{N_{sg}}}$$

The coherence modulus measures the signal phase shift coherence in time.

2.e Time Periodicity side effect

The fact that, for the discrete-time Fourier analysis, the finite discrete-time signal is considered as periodic have side effects on the circular signal autocorrelation.

$$C_a[m,1]=\frac{1}{N_{Ft}}\sum_{k=1}^{N_{Ft}} a_k^* a_{mod(k+m-1,N_{Ft})}$$

If the signal is a random signal with maximum time correlation smaller than $\frac{N_{Ft}}{2}t_s$, the side effects are limited : the role of terms due to the signal periodicity is small compared to the rest of the signal.

If the signal is coherent for times larger than $\frac{N_{Ft}}{2}t_s$, the side effects might be important.

For example, we consider a signal as a single frequency signal. If the signal frequency is a sampling frequency divider, there is no side effect. If not, the discrete-time signal Fourier transform shows power for all frequencies, especially the ones close to the signal frequency.

Windowing and overlapping methods

This side effect can be partially corrected if the segment limit is less abrupt. Alternative methods use different data weighting functions (also known as window functions) and data segments overlapping.

3.3 Discrete Linear System and Filtering

Digital filtering consists of applying a discrete linear system to a digital signal. In order to describe the behavior of these linear systems, we introduce the z-transform.

3.a Z-transform

For real signals, the Laplace transform is:

$$\mathring{s}(p)=s(t)e^{-pt}$$

For the sampled signals, we introduce the equivalent notion of z-transform:

$$X(z)=\sum_0^{+\infty} x_k z^{-k}$$

where the equivalence is obtained by setting :

$$z=e^{-pt_s}$$

This z-transform approximates the Fourier transform at the frequency f in the case where :

$$z=e^{i2\pi f t_s}$$

This relation applies the Fourier transform for a sampled signal periodic with period f_s .

The z-transform has properties similar to the Fourier transform:

- the z-transform is linear;
- the z-transform of a product is the convolution product of the z-transforms.

For time discrete signals, the convolution product is simply written:

$$(h*x)_n=\sum_{k=-\infty}^{+\infty} h_k x_{n-k}$$

3.b Discrete Linear System (DLS)

A Discrete Linear System h is a transformation of a time series x_n into another time series y_n .

An DLS has the property of being linear and time-invariant. It is characterized by its origin impulse response $u_n = \delta_{n,0}$. This response is h_n :

$$SLD(u)_n = h_n$$

By the DLS time invariance property, the response to an impulse at an index k , $u'_n = \delta_{n,k}$, is :

$$SLD(u')_n = h_{n-k}$$

Using DLS linearity property, the response to any signal x_n , is :

$$y_n = \sum_{k=-\infty}^{+\infty} h_{n-k}\, x_k$$

The DLS is characterized by the origin impulse response h_n . The DLS output is the discrete convolution product input with h_n .

Causal DLS

An DLS is causal if it verifies :

$$\forall k < 0,\ h_k = 0$$

Stable DLS

A DLS is stable if for any bounded input, the output is bounded :

$$\forall x | \sum_{n=-\infty}^{+\infty} |x_n| < +\infty \ , \quad \sum_{n=-\infty}^{+\infty} |y_n| < +\infty$$

The condition holds if and only if :

$$\sum_{n=-\infty}^{+\infty} |h_k| < +\infty$$

DLS z-transfer function

As the output of an DLS is the convolution product of the input by the transfer series, the z-transform of the output is the product of the z-transforms of the input by the transfer function:

$$Y(z) = H(z)X(z)$$

In the frequency domain, this means that the value of the function for a frequency f , $H(e^{i2\pi f t_s})$, describes the amplitude attenuation, and the phase shift of the output signal with respect to the input signal.

3.c DLS finite and Infinite Impulse Response

DLS Infinite Impulse Response (IIR)

In order to build a system with infinite impulse response while using a finite number of components, the expression of the output y_n , depends on input values

x_{n-k} , but also output values y_{n-k} :

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{N} b_k y_{n-k}$$

The largest index N is the order of the system (with $a_N \neq 0$ or $b_N \neq 0$).

In the z space, this relation is:

$$Y(z) = \sum_{k=0}^{N} a_k z^{-k} X(z) - \sum_{k=1}^{N} b_k z^{-k} Y(z)$$

The transfer function is :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 + \sum_{k=1}^{N} b_k z^{-k}}$$

The filter is stable if the poles of the transfer function have a modulus strictly less than 1.

Due to the denominator, the phase response of the system is non-linear: the response can be very variable around certain frequencies close to the denominator zeros.

DLS Finite impulse response (FIR)

For a finite impulse response SLD, the response can be expressed using a finite number of input values :

$$y_n = \sum_{k=0}^{N} a_k x_{n-k}$$

The transfer function is simply :

$$H(z) = \sum_{k=0}^{N} a_k z^{-k}$$

A finite impulse response DLS is always stable. The phase response is linear: there is no harmonic distortion of the signal. Its disadvantage is to require a much higher order than the infinite impulse response DLS to obtain a comparable frequency response.

4 Continuous random variables

In this part, we study continuous random variable : the variable has no time dependence.

The first section recalls some definitions and properties concerning continuous random variables.

The second section deals with random variable pairs.

4.1 Distributions and moments

1.a Definition and examples

Definition

A continuous random variable X is defined by :

- a continuous range of values V (typically $\mathbb{R}$ or an interval of $\mathbb{R}$)
- a probability density (or distribution) P .

P is a function (or more generally distribution) from V to $\mathbb{R}^+$, integrable ($\int_{min(V)}^{x} P(x')dx'$ exists for all $x \in V$), and such that $\int_V P(x)dx = 1$.

$P(x)dx$ is the probability of finding the variable X between values x and $x + dx$.

In an improper way, we often identify the random variable X and one of these samples x .

Examples

The most common forms of probability density are as follows ($a \in \mathbb{R}$ et $b \in \mathbb{R}^{+*}$).

- The exponential distribution, set to $\mathbb{R}^+$:

$$P(x) = b\, e^{-bx}$$

- The Lorentz (or Cauchy) distribution, defined on $\mathbb{R}$:

$$P(x) = \frac{1}{\pi} \frac{b}{(x-a)^2 + b^2}$$

- The Gaussian (or normal) distribution, defined on $\mathbb{R}$:

$$P(x) = \frac{1}{\sqrt{2\pi b}} e^{-(x-a)^2/2b^2}$$

1.b Means and Moments

In the following, we consider the probability distributions defined for the reals.

We define, provided that the following integrals converge, the following values:

- the mean (or mean value) of the distribution μ_x :

$$\langle x\rangle = \int x\,P(x)\,dx = \mu_x$$

- more generally, the mean of the function f on the cast :

$$\langle f(x)\rangle = \int f(x)\,P(x)\,dx$$

- the m^{th} moment of the distribution (for $m \in \mathbb{N}^*$) :

$$\langle x^m\rangle = \int x^m\,P(x)\,dx$$

- the variance (or mean square deviation) :

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2$$

The integral defining the moments is not necessarily convergent: for example for the Lorentz distribution, only the first moment, the mean, exists. Variance, and subsequent moments, do not exist.

4.2 Characteristic function, moments and cumulants

2.a Characteristic function

The characteristic function of a random variable is the Fourier transform of its distribution:

$$G(k) = \int e^{i\,k\,x}\,P(x)\,dx = \langle e^{i\,k\,x}\rangle$$

The properties of the probability density induce the following properties for the characteristic function:

$$G(0) = 1$$

and

$$\forall k \in \mathbb{R}, |G(k)| \leq 1$$

For the Lorentz distribution, the characteristic function has the form:

$$G(k) = e^{i\,k\,a - b|k|}$$

For the Gaussian distribution:

$$G(k) = e^{i\,k\,a - b^2 k^2/2}$$

2.b Moments and Cumulants

The moment-generating function

If all the distribution moments exist, the characteristic function develops according to the series:

$$G(k) = 1 + i\,k\,\langle x\rangle - \frac{k^2}{2}\langle x^2\rangle \ldots + \frac{(ik)^m}{m!}\langle x^m\rangle \ldots$$

The characteristic function is said to be the moment generating function.

The Cumulant generating function

The cumulants κ_m of X are defined from the expansion of the logarithm of the

characteristic function of X :

$$\log G(k) = i\,k\,K_1 - \frac{k^2}{2} K_2 \ldots + \frac{(ik)^m}{m!} K_m \ldots$$

Cumulants are algebraic functions of moments. The first four are:

$$K_1 = \langle x \rangle = \mu_x$$

$$K_2 = \langle x^2 \rangle - \langle x \rangle^2 = \sigma_x^2$$

$$K_3 = \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3$$

$$K_4 = \langle x^4 \rangle - 4\langle x^3 \rangle \langle x \rangle - 3\langle x^2 \rangle^2 + 12\langle x^2 \rangle \langle x \rangle^2 - 6\langle x \rangle^4$$

The first cumulant is the mean.

The second cumulant is the variance, the square of the standard deviation, σ_x .

For the Gaussian distribution, by identifying the form of $\log G(k)$: $K_1 = a$, $K_2 = b$ and $K_m = 0$ if $m \geq 3$.

Unlike moments, the cumulants, from the second, do not depend on the centering of the function : if we shift the random variable x , $\hat{x} = x - a$, the probability distribution verifies : $\hat{P}(x) = P(x-a)$ .The characteristic function is then modified by : $\hat{G}(k) = G(k) e^{-i\,k\,a}$. Then only the first term of the series expansion of the characteristic function logarithm is modified : $\hat{K}_1 = K_1 - a$, and for any $m \geq 2$, $\hat{K}_m = K_m$.

The centered variable associated with x , often noted $\tilde{x}$, defined by $\tilde{x} = x - \langle x \rangle$ then simplifies the calculation of cumulants from the moments :

$$\widetilde{K}_1 = 0$$

$$K_2 = \widetilde{K}_2 = \langle \tilde{x}^2 \rangle = \sigma_x^2$$

$$K_3 = \widetilde{K}_3 = \langle \tilde{x}^3 \rangle$$

$$K_4 = \widetilde{K}_4 = \langle \tilde{x}^4 \rangle - 3\langle \tilde{x}^2 \rangle^2$$

The calculation of the cumulants for the centered variable, shows that the third cumulant is non-zero, provided that the distribution P is not symmetrical with respect to its mean

We define the coefficient of distribution skewness (asymmetry) by normalizing this third cumulant by the cube of the standard deviation :

$$\gamma_1 = \frac{K_3}{\sigma_x^3} = \frac{\langle \tilde{x}^3 \rangle}{\sigma_x^3}$$

The fourth cumulant is strictly positive if the values far from the mean value have a significant weight compared to those close, using the Gaussian distribution as reference. we define the fourth cumulant as the kurtosis (the distribution flattening) by normalizing this fourth cumulant by the squared variance:

$$\gamma_2 = \frac{K_4}{\sigma_x^4} = \frac{\langle \tilde{x}^4 \rangle}{\sigma_x^4} - 3$$

4.3 Coupled random variables

3.a Coupled random variables and independence

This section introduces the concept of n random real variables x_1 ... x_n , defined on n real intervals V_1 ... V_n . They are coupled by the probability distribution to n variables, $P(x_1,...,x_n)$, defined on $(V_1,...,V_n)$ to $\mathbb{R}^+$, integrable and verifying :

$$\int_{V_1}...\int_{V_n} P(x_1,...,x_n)\,dx_1...dx_n = 1 \quad .$$

The marginal distribution for each variable P_m is the probability distribution specific to each variable, knowing nothing about the value of the other variables. It is deduced from the global distribution by integration over all the other variables :

$$P_1(x_1) = \int_{V_2}...\int_{V_n} P(x_1,...,x_n)\,dx_2...dx_n$$

$$P_2(x_2) = \int_{V_1}\int_{V_3}...\int_{V_n} P(x_1,...,x_n)\,dx_1\,dx_3...dx_n \quad ...$$

Independence

The variables from x_1 to x_n are independent if :

$$\forall(x_1,...,x_n)\in(V_1,...,V_n) \qquad P(x_1,...,x_n) = \prod_{m=1}^{n} P_m(x_m)$$

Conditional probability

For simplicity, we limit ourselves to 2 variables. The concept easily generalizes to n variables.

The conditional probability, $P(x_1|x_2)$, is the probability of x_1 knowing the value of the second, x_2 . its value is:

$$P(x_1|x_2) = \frac{P(x_1,x_2)}{P_2(x_2)}$$

Conditional probability is verifies :

$$\int_{V_1} P(x_1|x_2)\,dx_1 = 1$$

The variables x_1 and x_2 are independent if and only if:

$$\forall(x_1,x_2)\in(V_1,V_2) \quad P(x_1|x_2) = P_1(x_1)$$

3.b Characteristic function, moments and cumulants

Moments

It is possible to define cross moments between variables:

$$\langle x_1^{m_1}...x_n^{m_n}\rangle = \int_{V_1}...\int_{V_n} x_1^{m_1}...x_n^{m_n} P(x_1,...,x_n)\,dx_1...dx_n$$

The Cauchy-Schwarz inequality implies that for any pair of random variables (if the order 2 moments exist):

$$\langle x_1 x_2\rangle^2 \le \langle x_1^2\rangle\langle x_2^2\rangle$$

The notion of characteristic function can be generalized :

$$G(k_1,\dots,k_n)=\int_{V_1}\dots\int_{V_n} e^{i k_1 x_1+\dots+ i k_n x_n} P(x_1,\dots,x_n)\,dx_1\dots dx_n$$

If the variables are independent, the characteristic function of the sum of independent variables is the product of the characteristic functions of each variable:

$$X_1\dots X_n \text{ independent } \Rightarrow\ G_{X_1+\dots+X_n}(k_1,\dots,k_n)=\prod_{m=1}^{n} G_{X_m}(k_m)$$

Cumulants

The cumulants can also be generalized to the multidimensional case, by developing in series the logarithm of the characteristic function on all the variables. Order m cumulants form a matrix of size n^m .

The first cumulants are numbered n : these are the mean values along each direction:

$$\mu_m=\langle x_m\rangle$$

The second cumulants form a matrix of size n^2 , the covariance matrix. Its elements are :

$$\sigma_{ml}=\langle x_m x_l\rangle-\langle x_m\rangle\langle x_l\rangle$$

This matrix is symmetric.

The diagonal terms are the variances specific to each variable. The off-diagonal terms are the co-variances.

If the covariance of 2 variables is zero, these variables are said to be uncorrelated. Independent variables are always uncorrelated. But the converse is not true: the condition of independence of the variables is much more restrictive than the notion of correlation of their values.

We introduce the notion of correlation coefficient :

$$\rho_{ml}=\frac{\sigma_{ml}}{\sigma_m\sigma_l}$$

Due to the Cauchy-Schwarz inequality:

$$|\rho_{ml}|\le 1$$

Multivariate Gaussian probability law

The Gaussian form of the probability law can be generalized to n dimensions. Lth vector $\bar{x}=(x_1,\dots,x_n)$ gathers all the random variables. The Gaussian form is determined by the vector of means $\bar{\mu}=(\mu_1,\dots,\mu_n)$ and by the covariance matrix, Σ , definite positive :

$$P(\bar{x})=\frac{1}{(2\pi)^{n/2}\,det(\Sigma)^{1/2}}\,e^{-(\bar{x}-\bar{\mu})\,.\,\Sigma^{-1}\,.\,(\bar{x}-\bar{\mu})/2}$$

The generating function of the multidimensional Gaussian law is written for the vector $\bar{k}=(k_1,\dots,k_n)$:

$$G(\bar{k})=e^{i\bar{k}\,.\,\bar{\mu}-\bar{k}\,.\,\Sigma\,.\,\bar{k}/2}$$

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