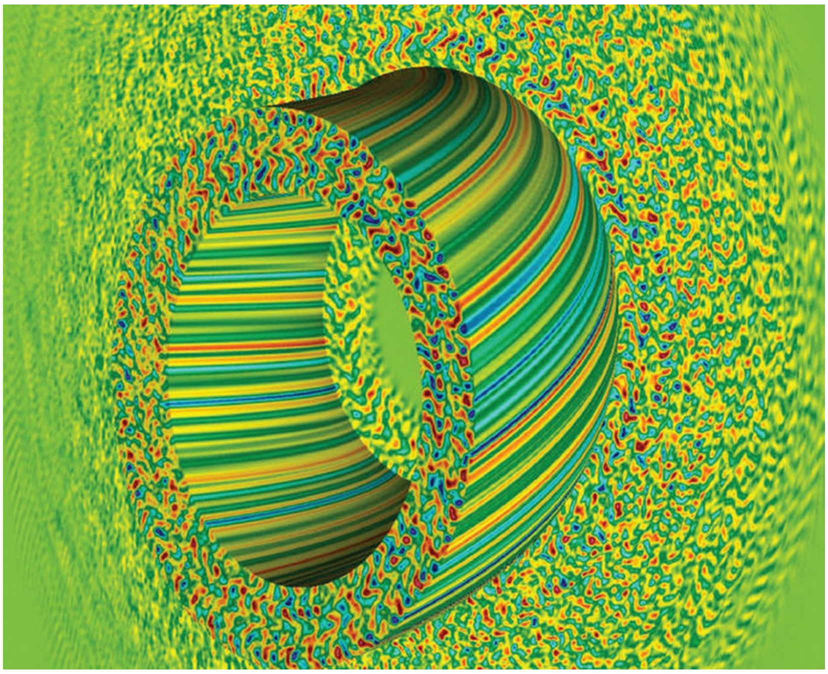
KINETIC MODEL Far
LINEAR ELECTROSTATIC INSTABILITIES IN A MAGNETIZED PLASMA

(Version 03/2024)



Simulation gyrocinétique ORB5 pour ITER

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Introduction

Contents

1 Linear kinetic model

1.1 Principle of linear kinetic model

1.a Particle distribution functions

The plasma distribution functions for each species α , $f_\alpha(\vec{r},\vec{v},t)$ depend on the particle position and velocity. For each species α , the mass is m_α and the charge q_α .

The integral of f_α on distribution velocity is the density of the species α :

$$n_\alpha(\vec{r},t)=\int d^3\vec{v}\, f_\alpha(\vec{r},\vec{v},t)$$

The first velocity distribution moment gives the local average velocity :

$$\langle\vec{v}_\alpha(\vec{r},t)\rangle=\frac{1}{n_\alpha(\vec{r},t)}\int d^3\vec{v}\,\vec{v}\, f_\alpha(\vec{r},\vec{v},t)$$

The second velocity distribution moment corresponds to the thermal velocity :

$$v_{T\alpha}^2(\vec{r},t)=\frac{1}{n_\alpha(\vec{r},t)}\int d\vec{v}_x(v_x-\langle\vec{v}_\alpha(\vec{r},t)\rangle)^2 f_\alpha(\vec{r},\vec{v},t)$$

The temperature is deduced from it :

$$k_B T_\alpha(\vec{r},t)=m_\alpha v_{T\alpha}^2$$

The integration of the kinetic energy along the 3 directions gives :

$$\frac{1}{n_\alpha(\vec{r},t)}\int d^3\vec{v}\,\frac{1}{2}m_\alpha(\vec{v}-\langle\vec{v}_\alpha(\vec{r},t)\rangle)^2 f_\alpha(\vec{r},\vec{v},t)=\frac{3}{2}k_B T_\alpha(\vec{r},t)$$

1.b Vlasov Equation

In order to establish the dynamics, we establish the analogy with the continuity equation for density. The conservation of matter implies that the time particle derivative d_t of the integrated density on any volume V verifies :

$$d_t\int_V d^3\vec{r}\, n_\alpha(\vec{r},t)=0$$

This property is expressed locally by :

$$\partial_t n_\alpha+\partial_t\vec{r}.\vec{\nabla}_{\vec{r}} n_\alpha=0$$

So :

$$\partial_t n_\alpha+\vec{v}.\vec{\nabla}_{\vec{r}} n_\alpha=0$$

For the species distribution function, the mass conservation property, is written :

$$d_t\int_{V_{\vec{r}}V_{\vec{v}}} d^3\vec{r}\, d^3v\, f_\alpha(\vec{r},\vec{v},t)=0$$

The differential expression ($d_t=\partial_t+\partial_t\vec{r}.\vec{\nabla}_{\vec{r}}+\partial_t\vec{v}.\vec{\nabla}_{\vec{v}}$) is :

$$\partial_t f_\alpha+\partial_t\vec{r}.\vec{\nabla}_{\vec{r}} f_\alpha+\partial_t\vec{v}.\vec{\nabla}_{\vec{v}} f_\alpha=0$$

So :

$$\partial_t f_\alpha+\vec{v}.\vec{\nabla}_{\vec{r}} f_\alpha+\vec{a}.\vec{\nabla}_{\vec{v}} f_\alpha=0 \qquad (1.1)$$

Where $\vec{a}$ is the particle acceleration due to electromagnetic forces :

$$\vec{a}=\frac{q_\alpha}{m_\alpha}\left(\vec{v}\wedge\vec{B}_z+\vec{E}\right)$$

This is the Vlasov equation. It assumes the absence of collisions and external sources of particles, by ionization, recombination or any chemical reactions.

The Vlasov equation is sufficient to describe the system dynamics.

1.c Poisson equation

We assume the electrostatic approximation: the magnetic field is imposed from the outside sources. The electric field derives from the plasma potential :

$$\vec{E}=-\vec{\nabla}\varphi$$

Using the Vlasov equation, it is possible to determine the distribution of each species as a function of the electric field. In order to determine the electric field, we use Poisson equation:

$$\vec{\nabla}.\vec{E}=\frac{1}{\varepsilon_0}\sum_\alpha q_\alpha\int d\vec{v}\, f_\alpha(\vec{r},\vec{v},t) \tag{1.2}$$

1.d Perturbative description

The plasma instabilities are studied with a perturbation method applied on an initial stable state.

The initial state is described by the distributions $f_{0\alpha}$ and the electric potential φ_0 . The perturbation is described by the perturbation on the distribution function superimposed on this initial state, $f_{1\alpha}$, and on the potential φ_1 .

$$f_\alpha(\vec{r},\vec{v},t)=f_{0\alpha}(\vec{r},\vec{v},t)+f_{1\alpha}(\vec{r},\vec{v},t)$$
$$\varphi(\vec{r},t)=\varphi_0(\vec{r},t)+\varphi_1(\vec{r},t)$$

This perturbation is described by a space and time Fourier mode, mainly along the direction $\vec{e}_y$, orthogonal to the density and potential gradients (along $\vec{e}_x$) and the magnetic field direction $\vec{e}_z$) The perturbation may also a component parallel to the magnetic field.

$$f_{1\alpha}(\vec{r},\vec{v},t)=\hat{f}_\alpha(\vec{k},\omega)e^{i(k_y y+k_z z-\omega t)}$$
$$\varphi_1(\vec{r},t)=\hat{\varphi}_1(\vec{k},\omega)e^{i(k_y y+k_z z-\omega t)}$$

The fluctuation amplitude is assumed to be small compared to the initial state :

$$\sqrt{\langle f_{1\alpha}^2\rangle}\ll\langle f_{0\alpha}(\vec{r},\vec{v},t)\rangle$$

We start by looking for the Vlasov equation solution for the 0th order :

$$\partial_t f_{0\alpha}+\vec{v}.\vec{\nabla}_{\vec{r}} f_{0\alpha}+\vec{a}_0.\vec{\nabla}_{\vec{v}} f_{0\alpha}=0$$

where the acceleration $\vec{a}_0$ excludes the effect of fluctuating electric field :

$$\vec{a}_0=\frac{q_\alpha}{m_\alpha}\left(\vec{v}\wedge\vec{B}_z-\vec{\nabla}\varphi_0\right)$$

We neglect the higher order terms. The Vlasov equation to the 1st order is:

$$\partial_t f_{1\alpha}+\vec{v}.\vec{\nabla}_{\vec{r}} f_{1\alpha}+\vec{a}_0.\vec{\nabla}_{\vec{v}} f_{1\alpha}=\frac{q_\alpha}{m_\alpha}\vec{\nabla}\varphi_1.\vec{\nabla}_{\vec{v}} f_0$$

The homogeneous part of this equation on the perturbation $f_{1\alpha}$ (left hand side) is identical to Vlasov equation to the 0^{th} order (it depends on $\vec{a}_0$ the particle acceleration with no perturbation). We can identify this homogeneous part as the particle derivative along solutions of the unperturbed equation (to the 0^{th} order). we will note this derivative d_{0t} .

The right hand side can be considered as a source term for the differential equation :

$$d_{0t} f_{1\alpha}=\frac{q_\alpha}{m_\alpha}\vec{\nabla}\varphi_1.\vec{\nabla}_{\vec{v}} f_0$$

The solution of this differential equation over time, along unperturbed trajectories, with the initial stable condition

$$f_{1\alpha}\underset{t\to-\infty}{\to}0\ ,$$

is :

$$\boxed{f_{1\alpha}(\vec{r}_0(t),\vec{v}_0(t),t)=\int_{-\infty}^{t} d\hat{t}\,\frac{q_\alpha}{m_\alpha}\vec{\nabla}\varphi_1(\vec{r}_0(\hat{t})).\vec{\nabla}_{\vec{v}} f_0(\vec{r}_0(\hat{t}),\vec{v}_0(\hat{t}),\hat{t})} \quad (1.3)$$

where $\vec{r}_0(\hat{t})$ and $\vec{v}_0(\hat{t})$ are the positions and velocities of the unperturbed particle trajectories.

We first have to evaluate the particle trajectory with no perturbation (0^{th} order).

This trajectory is necessary to evaluate $f_{1\alpha}$ in terms of φ_1 .

1.e Poisson equation and dielectric function

We can deduce from $f_{1\alpha}$ the density perturbation $n_{1\alpha}$:

$$n_{1\alpha}(\vec{r},t)=\int d\vec{v}\, f_{1\alpha}(\vec{r},\vec{v},t)$$

The system is closed by the Poisson equation.

Applied in the spatial Fourier space, to the 1^{st} order, the equation has the form :

$$k^2\hat{\varphi}_1=\frac{1}{\varepsilon_0}\sum_\alpha q_\alpha \hat{n}_{1\alpha}$$

This condition is written equivalently, in the form of the cancellation of the dielectric function:

$$\boxed{\epsilon(\vec{k},\omega)=1-\sum_\alpha \frac{q_\alpha \hat{n}_{1\alpha}}{\varepsilon_0 k^2\hat{\varphi}_1}=0} \quad (1.4)$$

For the electrostatic wave, the electric field is parallel to the wave vector $\vec{E}\|\vec{k}$. This relation introduces for each species α , the parallel susceptibility $\chi_{\|\alpha}$:

$$\chi_{\|\alpha}(\vec{k},\omega)=\frac{-q_\alpha \hat{n}_{1\alpha}}{\varepsilon_0 k^2\hat{\varphi}_1}$$

For the 1^{st} order development, $n_{1\alpha}$ is proportional to φ_1 : the parallel susceptibility is independent on the amplitude of the perturbation. It only depends on its wave vector $\vec{k}$ and its frequency ω .

The equation on the dielectric function makes it possible to establish the mode dispersion relation, the relationship between the wave vector $\vec{k}$ and the pulsation ω .

Actual frequency and growth rate

The 1st order density $n_{1\alpha}$ calculation may introduce an imaginary part for the dielectric function. In order to solve the complex equation $\epsilon(\vec{k},\omega)=0$, a 2 step approximation might be useful.

The approximated real part of the frequency ω_R is obtained by solving :

$$\boxed{\Re[\epsilon(\vec{k},\omega_R)]=0} \tag{1.5}$$

The growth rate of the instability γ is obtained using the resonant approximation : γ verifies :

$$\epsilon(\vec{k},\omega_R+i\gamma)=0$$

By carrying out a 1st series expansion around ω_R , this equation is :

$$\Re[\epsilon(\vec{k},\omega_R)]+i\gamma\partial_\omega\Re[\epsilon(\vec{k},\omega_R)]+i\Im[\epsilon(\vec{k},\omega_R)]-\gamma\partial_\omega\Im[\epsilon(\vec{k},\omega_R)]=0$$

The γ expression is deduced from the imaginary part of this equation.

$$\boxed{\gamma=\frac{-\Im[\epsilon(\vec{k},\omega_R)]}{\partial_\omega\Re[\epsilon(\vec{k},\omega_R)]}} \tag{1.6}$$

The last term of the former expression, $-\gamma\partial_\omega\Im[\epsilon(\vec{k},\omega_R)]$, is not necessarily zero : the result obtained is only an approximation of the solution. It will be valid if this term is negligible compared to the others.

1.2 Drift wave kinetic model

2.a The 0th order species distribution function

The geometry is slab, corresponding of local description of a toroidal plasma.

The magnetic field curvature and gradient are taken into account via the drift they imply particles.

The description uses a Cartesian frame of reference.

The magnetic field is constant and uniform along the direction $\vec{e}_z$, B_z :

$$\vec{B}=B_z\vec{e}_z$$

The species density gradient $\vec{\nabla}n_\alpha$ and the temperature gradient $\vec{\nabla}T_\alpha$ are along the direction $\vec{e}_x$:

$$\vec{\nabla}n_\alpha=d_x n_\alpha(x)\vec{e}_x$$

$$\vec{\nabla}T_\alpha=d_x T_\alpha(x)\vec{e}_x$$

The density n_α and the temperature T_α are uniform along the other 2 directions.

Particle motion invariants

For this system, the Lagrangian is given by:

$$L(\vec{r},\vec{v})=\frac{1}{2}m_\alpha v^2+q_\alpha\vec{v}.\vec{A}$$

where the vector potential $\vec{A}$ (in a specific gauge) takes the form :

$$\vec{A}=x\,B_z\vec{e}_y$$

As the Lagrangian is independent of time, the energy (the Hamiltonian) is conserved:

$$H=\vec{v}.\vec{\nabla}_{\vec{v}}L-L=\vec{v}.(m_\alpha\vec{v}+q_\alpha\vec{A})-L=\frac{1}{2}m_\alpha v^2$$

The energy is only the kinetic energy: the velocity modulus is conserved.

Since the Lagrangian is independent of particle coordinates x and y, the conjugate moments are preserved:

$$p_y=\partial_{v_y}L=m_\alpha v_y+q_\alpha B_z x$$

$$p_z=\partial_{v_z}L=m_\alpha v_z$$

The moment p_z conservation induces the velocity along the magnetic field is conserved (no force is applied on the particle in this direction). This implies that the velocity perpendicular to the magnetic field $v_\perp$ is conserved :

$$v_\perp=\sqrt{v_x^2+v_y^2}$$

The moment p_y conservation can be considered in a different way by dividing it by the constant $q_\alpha B_z$: this reveals the invariant X , which is in fact the trajectory guiding center position along $\vec{e}_x$:

$$\boxed{X=\frac{p_y}{q_\alpha B_z}=x+\frac{v_y}{\omega_{c\alpha}}} \tag{1.7}$$

where $\omega_{c\alpha}=\frac{q_\alpha B_z}{m_\alpha}$ is the cyclotron pulsation of the species α .

Initial distribution function

Thermodynamic equilibrium occurs quickly around a magnetic field line. But because of magnetic confinement, equilibrium is slow to settle from one magnetic field line to another. The behavior of the particles will not depend on their own position, which varies over time, but on the position of their guiding center, X , which is an invariant of the particle motion.

Due to other faster processes (ionization, drifts, etc.) which may appear, the axial profiles of the density $n_{g\alpha}(X)$ and temperature $T_\alpha(X)$ are not deduced from the equilibrium, but are constructed dynamically from all the processes: we consider them as external data.

We note $f_{g\alpha}(X,\vec{v})$ the particle distribution function, expressed as a function of the position of the guiding center :

$$\boxed{f_{g\alpha}(X,\vec{v})=\left(\frac{m_\alpha}{2\pi k_B T_\alpha(X)}\right)^{3/2}n_{g\alpha}(X)\,e^{\frac{-m_\alpha v^2}{2k_B T_\alpha(X)}}} \tag{1.8}$$

Around the guide center, the distribution of particle velocities is Maxwellian, with temperature $T_\alpha(X)$.

We assume the species density gradient lengths $L_{n\alpha}=\langle d_x \ln(n_{0\alpha})\rangle^{-1}$ and for the temperature $L_{T\alpha}=\langle d_x \ln(T_\alpha)\rangle^{-1}$ are large compared to the Larmor radius (i.e. the gradients are weak):

$\rho_{c\alpha} \ll L_{n\alpha}$ et $\rho_{c\alpha} \ll L_{T\alpha}$

The distribution function $f_\alpha(x, H)$ as a function of the particle position is calculated from the distribution function distribution as a function of the position of the particle guiding center position $X=x+\frac{v_y}{\omega_{c\alpha}}$:

$$f_{0\alpha}(x,\vec{v})=f_{g\alpha}(x+\frac{v_y}{\omega_{c\alpha}},\vec{v})$$

The value is approximated by a perturbative expansion on the distribution function depending on the position of the guiding center. The particle distribution function as a function of the position of the particle is written:

$$\boxed{f_{0\alpha}(x,\vec{v})=f_{g\alpha}(x,\vec{v})+\frac{v_y}{\omega_{c\alpha}}\partial_X f_{g\alpha}(x,\vec{v})} \tag{1.9}$$

The relationship between both distribution functions depends on the density and temperature gradients:

$$\boxed{\partial_x f_{g\alpha}(x,\vec{v})=\left[d_X \ln(n_{g\alpha})+d_X \ln(T_\alpha)\left(\frac{\frac{1}{2}m_\alpha v^2}{k_B T_\alpha}-\frac{3}{2}\right)\right]f_{g\alpha}(x,\vec{v})} \tag{1.10}$$

Because of these gradients, the particle distribution function is no longer Maxwellian.

Particle density

The α species density is :

$$n_{0\alpha}(x)=\int d^3\vec{v}\, f_{0\alpha}(x,H)$$

or :

$$n_{0\alpha}(x)=\int d^3\vec{v}\, f_{g\alpha}(x,\vec{v})+\int d^3\vec{v}\,\partial_x f_{g\alpha}(x,\vec{v})\frac{v_y}{\omega_{c\alpha}}$$

Since the built-in function is antisymmetric in velocity component v_y , the 2nd integrated term is zero.

The particle density is identical to the density profile given as a function of the position of the guiding center:

$$\boxed{n_{0\alpha}(x)=n_{g\alpha}(x)} \tag{1.11}$$

Average drift velocity

Even if each particle has a cyclotron motion with no drift, A collective plasma drift appears when the each species α velocity is averaged locally.

The average velocity depends on the position.

$$\langle \vec{v_{0\alpha}}(x)\rangle = \frac{1}{n_{0\alpha}} \int d^3\vec{v}\,\vec{v}\, f_{0\alpha}(x, H)$$

or :

$$\langle \vec{v_{0\alpha}}(x)\rangle = \frac{1}{n_{0\alpha}} \int d\vec{v}^3\vec{v}\, f_{g\alpha}(X, \vec{v}) + \frac{1}{n_{0\alpha}} \int d\vec{v}^3\vec{v}\,\partial_X f_{g\alpha}(X, \vec{v}) \frac{v_y}{\omega_{c\alpha}}$$

Because the first integrated function is antisymmetric with respect to the velocity, the first integrated term is zero.

Concerning the second term, the function inside the integration along $\vec{e_x}$ et $\vec{e_z}$ is also antisymmetric. The remaining integration is:

$$\langle \vec{v_{0\alpha}}(x)\rangle = \frac{1}{n_{0\alpha}} \int d\,v_y \left[d_X \ln(n_{g\alpha}) + d_X \ln(T_\alpha) \left(\frac{\frac{1}{2} m_\alpha v^2}{k_B T_\alpha} - \frac{3}{2} \right) \right] f_{g\alpha}(x, \vec{v}) \frac{v_y^2}{\omega_{c\alpha}} \vec{e_y}$$

For a Maxwellian:

$$\frac{1}{n_\alpha} \int d\,v_y v_y^2 f_\alpha = v_{T\alpha}^2$$

and

$$\frac{1}{n_\alpha} \int d\,v_y v_y^4 f_\alpha = 3\,v_{T\alpha}^4$$

we obtain for the average velocity:

$$\langle \vec{v_{0\alpha}}(x)\rangle = \left[d_X \ln(n_{0\alpha}) + d_X \ln(T_\alpha) \right] \frac{k_B T_\alpha}{q_\alpha B_z} \vec{e_y}$$

This velocity can be considered at the mesoscopic fluid scale.

The pressure for the species α is:

$$P_\alpha = n_{0\alpha} k_B T_\alpha$$

The pressure gradient is then :

$$\nabla P_\alpha = n_{0\alpha} k_B T_\alpha (d_X \ln(n_{0\alpha}) + d_X \ln(T_\alpha)) \vec{e_x}$$

The drift velocity is only a function of the pressure gradient :

$$\boxed{\langle \vec{v}_{0\alpha}(x)\rangle = \frac{-\vec{\nabla} P_\alpha \wedge \vec{B_z}}{q_\alpha n_{0\alpha} B_z^2}} \tag{1.12}$$

This velocity, defined from the pressure gradient, is the diamagnetic drift for the species α . Unlike other drifts ($\vec{E}_0 \wedge \vec{B}$ or $\vec{\nabla} B \wedge \vec{B}$) , this drift does not concern the guiding center, but appears as a collective effect.

The sign depends on the species charge sign: the ion and electron diamagnetic drifts have opposite directions.

In the case of a density gradient, this drift is due to the fact that for a position of a given particle position x , the number of particles with a positive velocity v_y is different from the number of particles having a negative velocity v_y , since the guiding center position X is different for positive and negative velocities v_y and since the guiding center density differs because of the density gradient.

Drifts specific to additional forces

Other forces along x can be taken into account :

the force from a uniform electric field E_{x0} or F_B , the force analogous to the drift due to the toroidal form of the magnetic field :

$$\vec{F}_B = m_\alpha \left(\frac{1}{2} v_\perp^2 + v_z^2 \right) \frac{1}{R_c} \vec{e}_x$$

where R_c is the major radius of the torus.

The particle distribution function is then modified. These forces modify the expression of the Hamiltonian for the guide centers:

$$f_{g\alpha}(X, \vec{v})^* = \left(\frac{m_\alpha}{2\pi k_B T_0(X)} \right)^{3/2} n_{g\alpha}(X) e^{\frac{-m_\alpha v^2}{2 k_B T_\alpha(X)}} e^{\frac{(F_B + q_\alpha E_{x0}) X}{k_B T_\alpha(X)}}$$

This modification should have an impact on the particle distribution in the direction x . But as this direction is perpendicular to the magnetic field lines, this effect is slow: we will assume that this effect is taken into account in the axial density and temperature profiles.

These forces nevertheless have an effect on the trajectory of the particles around the guiding center. The acceleration occurring in Vlasov equation to the 0th order is not only :

$$\vec{a}_0 = \frac{q_\alpha}{m_\alpha} (\vec{v} \wedge \vec{B}_z)$$

but it also includes :

$$\vec{a}_0 = \frac{q_\alpha}{m_\alpha} (\vec{v} \wedge \vec{B}_z + \vec{E}_{0x}) + \frac{1}{m_\alpha} \vec{F}_B \quad .$$

The reference frame is changed using the following velocity with the relation :

$$\vec{v} = \vec{u} + \vec{v}_{E0} + \vec{v}_{B\alpha}$$

The additional velocities are the drift velocities associated with these forces in the direction perpendicular to the magnetic field and the electric field $\vec{e}_y$:

$$\vec{v_{E0}} = \frac{\vec{E}_{0x} \wedge \vec{B}_z}{B_z^2} = \frac{-E_{0x}}{B_z} \vec{e}_y$$

and

$$\vec{v_{B\alpha}} = \frac{\vec{F}_B \wedge \vec{B}_z}{q_\alpha B_z^2} = \frac{-F_B}{q_\alpha B_z} \vec{e}_y$$

We note the sum of these drift velocities, $v_{f\alpha}$:

$$\vec{v}_{E_0} + \vec{v}_{F_B} = v_{f\alpha} \vec{e}_y$$

with :

$$v_{f\alpha} = \frac{-E_{0x}}{B_z} - \frac{F_B}{q_\alpha B_z}$$

In the moving frame, even if it is in uniform motion with respect to the laboratory frame, the acceleration is modified because of the dependence of the Lorentz force on the velocity.

In this frame, the acceleration is:

$$\vec{a}_0=\frac{q_\alpha}{m_\alpha}(\vec{u}_0\wedge\vec{B}_z)$$

In the repository associated with $\vec{u}_0$, The guiding center distribution is the same exposed for the case without additional forces.

Relative to the laboratory reference frame, the distribution according to the position of the guide centers is deduced:

$$\boxed{f_{g\alpha}(X,\vec{v})=\left(\frac{m_\alpha}{2\pi k_B T_\alpha(X)}\right)^{3/2} n_{g\alpha}(X)\, e^{\frac{-m_\alpha}{2k_B T_\alpha(X)}\left[v_x^2+(v_y-v_{f\alpha})^2+v_z^2\right]}} \quad (1.13)$$

For the approximation to be valid, it is necessary that the variations in potential energy specific to the force $q_\alpha\vec{E}_0+\vec{F}_B$ on the scale of the Larmor radius are small compared to the thermal energy. In this case, the drift velocity $v_{f\alpha}$ specific to these forces, is small compared to the species thermal velocity $v_{T\alpha}$:

$$\frac{|q_\alpha\vec{E}_0+F_B|\rho_{c\alpha}}{k_B T_\alpha}=\frac{|v_{f\alpha}|}{v_{T\alpha}}\ll 1$$

In this frame, the (invariant) particle guiding center position X is:

$$\boxed{X=\frac{p_y}{q_\alpha B_z}=x+\frac{v_y-v_{f\alpha}}{\omega_{c\alpha}}} \quad (1.14)$$

The α particle distribution function is deduced from the guiding center distribution:

$$\boxed{f_{0\alpha}(x,H)=f_{g\alpha}(x,\vec{v})+\partial_X f_{g\alpha}(x,\vec{v})\frac{v_y-v_{f\alpha}}{\omega_{c\alpha}}} \quad (1.15)$$

with:

$$\boxed{\partial_x f_{g\alpha}(x,\vec{v})=\left[d_X\ln(n_{g\alpha})+d_X\ln(T_\alpha)\left(\frac{\frac{1}{2}m_\alpha\left[v_x^2+(v_y-v_{f\alpha})^2+v_z^2\right]}{k_B T_\alpha}-\frac{3}{2}\right)\right]f_{g\alpha}(x,\vec{v})}$$

Each species particle density has the same expression as the particle guiding center position:

$$\boxed{n_{0\alpha}(x)=n_{g\alpha}(x)}$$

and the particle drift velocity is:

$$v_{f\alpha}=\frac{-E_{0x}}{B_z}-\frac{F_B}{q_\alpha B_z}$$

Concerning the drift velocities, $\vec{E}\wedge\vec{B}$ drift and $\vec{F}_B\wedge\vec{B}$ drift are added to the diamagnetic drift.

The species average velocity is:

$$\boxed{\langle\vec{v}_{0\alpha}(x)\rangle=-\frac{\vec{\nabla}P_\alpha\wedge\vec{B}_z}{q_\alpha n_{0\alpha}B_z^2}+\frac{\vec{E}_{0x}\wedge\vec{B}_z}{B_z^2}+\frac{\vec{F}_B\wedge\vec{B}_z}{q_\alpha B_z^2}} \quad (1.16)$$

2.b The 0th order particle trajectory

In the moving reference frame associated with $\vec{u}_0$ velocity, shifted from drift velocities due to external forces:

$$\vec{v}_0 = \vec{u}_0 + v_{f\alpha}\vec{e}_y$$

the particle acceleration is:

$$\vec{a}_0 = \frac{q_\alpha}{m_\alpha}(\vec{u}_0 \wedge \vec{B}_z)$$

In this reference frame, the particle motion is a cyclotron motion (t_i is the initial time, t is the current time) :

$$u_{0x}(t) = u_{0\perp}\cos[\omega_{c\alpha}(t-t_i)+\psi_0]$$
$$u_y(t) = u_{0\perp}\sin[\omega_{c\alpha}(t-t_i)+\psi_0]$$
$$u_z(\dot{t}) = u_{0z}$$

where $u_{0\perp}$ is the velocity perpendicular to the magnetic field :

$$u_{0\perp} = \sqrt{u_{0x}^2 + u_{0y}^2}$$

ψ_0 is the cyclotron perpendicular velocity phase at the initial time t .

Back in the laboratory reference frame, the particle velocity components are:

$$v_x(t) = u_{0\perp}\cos[\omega_{c\alpha}(t-t_i)+\psi_0]$$
$$v_y(t) = u_{0\perp}\sin[\omega_{c\alpha}(t-t_i)+\psi_0]+v_{f\alpha}$$
$$v_z(t) = v_{0z}$$

The diamagnetic drift does not play a role here, because it is a collective effect which does not affect the particle individual trajectories.

The particle perpendicular velocity $u_{0\perp}$ is :

$$u_{0\perp} = \sqrt{v_{0x}^2 + (v_{0y} - v_{f\alpha})^2}$$

The particle position in the laboratory reference frame is calculated from the particle velocity components:

$$\boxed{x_0(t) = \frac{u_{0\perp}}{\omega_{c\alpha}}\left[\sin[\omega_{c\alpha}(t-t_i)+\psi_0] - \sin\psi_0\right] + x(t_i)} \qquad (1.17)$$

$$\boxed{y_0(t) = \frac{u_{0\perp}}{\omega_{c\alpha}}\left[-\cos[\omega_{c\alpha}(t-t_i)+\psi_0] + \cos\psi_0\right] + v_{f\alpha}(t-t_i) + y(t_i)} \qquad (1.18)$$

$$\boxed{z_0(t) = v_{0z}(t-t_i) + z_0(t_i)} \qquad (1.19)$$

The expression of the particle trajectory will be necessary to evaluate the perturbed distribution function.

2.c The 1st order species distribution function

Integration following undisturbed trajectories

As seen above, the perturbed distribution function is calculated by the integral of the 1st order electric field, along the unperturbed particle trajectories:

$$f_{1\alpha}(\vec{r}_0(t),\vec{v}_0(t),t)=\int_{-\infty}^{t} d\hat{t}\frac{q_\alpha}{m_\alpha}\vec{\nabla}\varphi_1[\vec{r}_0(\hat{t})].\vec{\nabla}_{\vec{v}} f_0[\vec{r}_0(\hat{t}),\vec{v}_0(\hat{t}),\hat{t}]$$

To simplify the calculations, we only consider the case of a uniform temperature profile for each species:

$$T_\alpha(x)=T_\alpha$$

The temperature gradients are equal to zero :

$$d_X \ln(T_\alpha)=0$$

The expression of the distribution at the 0th order is simplified :

$$f_{0\alpha}=\left(1+d_x\ln(n_{0\alpha})\frac{v_y-v_{f\alpha}}{\omega_{c\alpha}}\right)f_{g\alpha}$$

with

$$f_{g\alpha}(X,\vec{v})=\left(\frac{m_\alpha}{2\pi k_B T_\alpha}\right)^{3/2} n_{g\alpha}(X)\, e^{\frac{-m_\alpha}{2k_B T_\alpha}(\vec{v}-\vec{v}_{E0}-\vec{v}_{B\alpha})^2}$$

The distribution gradient for velocity is :

$$\vec{\nabla}_{\vec{v}} f_{0\alpha}=\left[d_x\ln(n_{0\alpha})\frac{1}{\omega_{c\alpha}}\vec{e}_y-\frac{m_\alpha}{k_B T_\alpha}(\vec{v}-v_{f\alpha}\vec{e}_y)\right]f_{0\alpha}$$

The source term for the 1st order Vlasov equation is for a Fourier mode $(\vec{k},\omega)$:

$$\vec{\nabla}\varphi_1.\vec{\nabla}_{\vec{v}} f_0=i\left[\frac{k_y}{\omega_{c\alpha}}d_x\ln(n_{0\alpha})-\frac{m_\alpha}{k_B T_\alpha}\vec{k}.(\vec{v}-v_{f\alpha}\vec{e}_y)\right]f_{0\alpha}\varphi_1$$

or

$$\vec{\nabla}\varphi_1.\vec{\nabla}_{\vec{v}} f_0=\left[i k_y(v_{n\alpha}+v_{f\alpha})-i\vec{k}.\vec{v}\right]\frac{1}{v_{th\alpha}^2} f_{0\alpha}\hat{\varphi}_1 e^{i(\vec{k}.\vec{r}-\omega t)}$$

the species diamagnetic velocity with no temperature gradient is:

$$\boxed{v_{n\alpha}=d_X\ln(n_{0\alpha})\frac{k_B T_\alpha}{q_\alpha B_z}} \tag{1.20}$$

The diamagnetic drift is added to the drifts due to external forces:

$$\boxed{v_{d\alpha}=v_{n\alpha}+v_{f\alpha}} \tag{1.21}$$

This source term must be integrated along the particle unperturbed trajectories (position $\vec{r}_0(\hat{t})$, velocity $\vec{v}_0(\hat{t})=d_{\hat{t}}\vec{r}_0(\hat{t})$).

These trajectories satisfy Vlasov equation:

$$d_t f_{0\alpha}(\vec{r}_0,\vec{v}_0,t)=0$$

This implies:

$$d_t[f_{0\alpha}\hat{\varphi}_1 e^{i(\vec{k}.\vec{r}-\omega t)}]=i(\vec{k}.\vec{v}_0-\omega)f_{0\alpha}\hat{\varphi}_1 e^{i(\vec{k}.\vec{r}-\omega t)}$$

The source term is:

$$\nabla\varphi_1.\nabla_{\vec{v}} f_0=\left[i k_y v_{d\alpha}-i\omega-d_t\right]\frac{m_\alpha}{k_B T_\alpha} f_{0\alpha}\hat{\varphi}_1 e^{i(\vec{k}.\vec{r}-\omega t)}$$

This expression is introduced into the expression of the distribution at the 1st order:

$$f_{1\alpha}\left(\vec{r}_0(t),\vec{v}_0(t),t\right)=\int_{-\infty}^{t} d\hat{t}\left[i k_y v_{d\alpha}-i\omega-d_{\hat{t}}\right]\frac{q_\alpha}{k_B T_\alpha} f_{0\alpha}\hat{\varphi}_1 e^{i\left(\vec{k}.\vec{r}_0(\hat{t})-\omega\hat{t}\right)}$$

The expression is integrated by part:

$$f_{1\alpha}\left(\vec{r}_0(t),\vec{v}_0(t),t\right)=-\left[1-i\left(k_y v_{d\alpha}-\omega\right)\int_{-\infty}^{t} d\hat{t}\, e^{i\left[\vec{k}.\left(\vec{r}_0(\hat{t})-\vec{r}\right)-\omega(\hat{t}-t)\right]}\right]\frac{q_\alpha}{k_B T_\alpha} f_{0\alpha}\hat{\varphi}_1$$

Phase integration along particle trajectories

The phase integration is done along the unperturbed trajectories:

$$\phi(t)=\int_{-\infty}^{t} d\hat{t}\, e^{i\left[\vec{k}.\left(\vec{r}_0(\hat{t})-\vec{r}_0(t)\right)-\omega(\hat{t}-t)\right]}$$

$k_\perp$ and θ are the wave number and the polar angle of the wave vector $\vec{k}$ in the plan $\left(\vec{e}_x,\vec{e}_y\right)$ perpendicular to the magnetic field:

$$\vec{k}=k_\perp\cos\theta\,\vec{e}_x+k_\perp\sin\theta\,\vec{e}_y+k_z\vec{e}_z$$

The particle unperturbed trajectory is:

$$x_0(t)=\frac{u_{0\perp}}{\omega_{c\alpha}}\left[\sin\left[\omega_{c\alpha}(t-t_i)+\psi_0\right]-\sin\psi_0\right]+x(t_i)$$

$$y_0(t)=\frac{u_{0\perp}}{\omega_{c\alpha}}\left[-\cos\left[\omega_{c\alpha}(t-t_i)+\psi_0\right]+\cos\psi_0\right]+v_{f\alpha}(t-t_i)+y(t_i)$$

$$z_0(t)=v_{0z}(t-t_i)+z_0(t_i)$$

the first term of the integrated phase is:

$$\vec{k}.\left(\vec{r}_0(\hat{t})-\vec{r}_0(t)\right)=\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\left[\sin\left[\omega_{c\alpha}(\hat{t}-t)+\psi_0+\theta\right]-\sin\left(\psi_0+\theta\right)\right]+k_y v_{f\alpha}(t'-t)+k_z v_{0z}(\hat{t}-t)$$

The integrated phase is:

$$\phi(t)=\int_{-\infty}^{0} d\tau\, e^{i\left[\frac{k_{0\perp}u_\perp}{\omega_{c\alpha}}\left[\sin\left[\omega_{c\alpha}\tau+\psi_0+\theta\right]-\sin\left(\psi_0+\theta\right)\right]+k_y v_{f\alpha}\tau+k_z v_{0z}\tau\right]}$$

This phase oscillates at the particle gyration frequency around the guiding center. We use the property of the sine function phase decomposition in a sum on the harmonics weighted by the 1st Bessel functions J_n :

$$e^{iz\sin\theta}=\sum_{n=-\infty}^{+\infty} J_n(z)e^{in\theta}$$

We apply this transformation to the 2 phases containing sinuses:

$$e^{i\left[\frac{k_{0\perp}u_{0\perp}}{\omega_{c\alpha}}\sin\left(\omega_{c\alpha}\tau+\psi_0+\theta\right)\right]}=\sum_{n=-\infty}^{+\infty} J_n\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)e^{in\left(\omega_{c\alpha}\tau+\psi_0+\theta\right)}$$

$$e^{-i\left[\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\sin\left(\psi_0+\theta\right)\right]}=\sum_{\hat{n}=-\infty}^{+\infty} J_{\hat{n}}\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)e^{-i\hat{n}\left(\psi_0+\theta\right)}$$

By applying these 2 transformations, the phases are linear functions of τ :

$$\phi(t)=\int_{-\infty}^{0} d\tau \sum_{n,n'=-\infty}^{+\infty} J_n\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)J_{\hat{n}}\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)e^{i(n-n')\left(\psi_0+\theta\right)}\,e^{i\left[n\omega_{c\alpha}+k_y v_{f\alpha}+k_z v_z-\omega\right]\tau}$$

The integration on τ is possible:

$$\phi(t)=\sum_{n,\hat{n}=-\infty}^{\infty}\frac{J_n\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)J_{\hat{n}}\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)e^{i(n-\hat{n})(\psi_0+\theta)}}{i(n\omega_{c\alpha}+k_y v_{de\alpha}+k_z v_{0z}-\omega)}$$

The result introduces a denominator. This denominator is equal to zero if the movement of the particle parallel to the magnetic field is resonant with the wave. This singularity will be treated later.

The distribution function at the 1st order expression is:

$$\boxed{f_{1\alpha}=-\left[1+(\omega-k_y v_{d\alpha})\sum_{n,\hat{n}=-\infty}^{+\infty}\frac{J_n\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)J_{\hat{n}}\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)e^{i(n-\hat{n})(\psi_0+\theta)}}{n\omega_{c\alpha}+k_y v_{f\alpha}+k_z v_{0z}-\omega}\right]\frac{q_\alpha}{k_B T_\alpha}f_{0\alpha}\hat{\varphi}_1} \qquad (1.22)$$

2.d The 1st order species density

The dispersion relation is obtained by closing the problem with the Poisson equation which involves the density of the species α :

$$\epsilon(\vec{k},\omega)=1-\sum_\alpha\frac{q_\alpha\hat{n}_{1\alpha}}{\varepsilon_0 k^2\hat{\varphi}_1}=0$$

The density fluctuations are the integration of the distribution function on the velocity:

$$n_{1\alpha}(\vec{r},t)=\int d^3\vec{v}\, f_{1\alpha}(\vec{r},\vec{v},t)$$

$f_{1\alpha}$ expression depends on $f_{0\alpha}$: to simplify the calculations we neglect the effect of the density gradient in the expression de $f_{0\alpha}$: we use the expression of the distribution function attached to the particle guiding center positions: for the calculation of the density at the 0th order, the correction by the density gradient did not change the result. We will assume that this effect at the 1st order is only marginal:

$$f_{0\alpha}(x,\vec{v})\sim\left(\frac{m_\alpha}{2\pi k_B T_\alpha}\right)^{3/2}n_{0\alpha}(x)\,e^{\frac{-m_\alpha}{2k_B T_\alpha}\left[u_{0\perp}^2+v_z^2\right]}$$

Since $f_{0\alpha}$ and $f_{1\alpha}$ as velocity functions are expressed in the cylindrical coordinates of the velocity (in the moving reference frame), The distribution function will be integrated on the velocity with these cylindrical coordinates $(u_\perp,\psi,v_z)$:

$$n_{1\alpha}(\vec{r},t)=\int_{-\infty}^{+\infty}dv_{0z}\int_0^\infty du_{0\perp}u_{0\perp}\int_0^{2\pi}d\psi_0 f_{1\alpha}(\vec{r},u_{0\perp},\psi_0,v_z)$$

Integration on the velocity phase ψ_0

Since $f_{0\alpha}$ is independent of ψ_0 , the integration on ψ_0 is quite simple :

$$\int_0^{2\pi}d\psi_0 f_{1\alpha}=-\left[1+(\omega-k_y v_{d\alpha})\sum_{n,\hat{n}=-\infty}^{\infty}\frac{J_n\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)J_{\hat{n}}\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)\int_0^{2\pi}d\psi_0 e^{i(n-\hat{n})(\psi_0+\theta)}}{n\omega_{c\alpha}+k_y v_{f\alpha}+k_z v_{0z}-\omega}\right]\frac{q_\alpha}{k_B T_\alpha}f_{0\alpha}\hat{\varphi}_1$$

As :

$$\forall a\neq 0\;\int_0^{2\pi}d\psi_0 e^{ia\psi_0}=0$$

the integration on ψ_0 of the factors $e^{i(n-\hat{n})(\psi_0+\theta)}$ present in the expression of

$f_{1\alpha}$, are equal to zero for the non-diagonal terms of the summations on n and n' (i.e. $n \neq n'$).

The summation is reduced to the diagonal terms ($n=\hat{n}$), for which the expression does not depend on ψ_0 . The integration on the phase ψ_0 is simplified :

$$\int_0^{2\pi} d\psi_0 f_{1\alpha} = -2\pi \left[1 + (\omega - k_y v_{d\alpha}) \sum_{n=-\infty}^{+\infty} \frac{J_n^2\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)}{n\omega_{c\alpha} + k_y v_{f\alpha} + k_z v_{0z} - \omega} \right] \frac{q_\alpha}{k_B T_\alpha} f_{0\alpha} \hat{\varphi}_1$$

Perpendicular velocity $u_{0\perp}$ and finite Larmor radius $\rho_{cth\alpha}$

The expression of $f_{0\alpha}$ to be able to integrate on $u_{0\perp}$:

$$\int_0^\infty du_{0\perp} u_{0\perp} \int_0^{2\pi} d\psi_0 f_{1\alpha} =$$

$$-\left[1 + (\omega - k_y v_{d\alpha}) \sum_{n=-\infty}^{\infty} \frac{\int_0^\infty du_{0\perp} u_{0\perp} e^{\frac{-u_\perp^2}{2 v_{T\alpha}^2}} J_n^2\left(\frac{k_\perp u_{0\perp}}{\omega_{c\alpha}}\right)}{n\omega_{c\alpha} + k_y v_{f\alpha} + k_z v_{0z} - \omega} \right] \frac{n_{0\alpha}(x) q_\alpha}{k_B T_\alpha} \frac{e^{\frac{-v_z^2}{2 v_{T\alpha}^2}}}{\sqrt{2\pi} v_{T\alpha}} \hat{\varphi}_1$$

The integration on $u_\perp$ involves modified Bessel functions $I_n(r)$ who are defined for $r \geq 0$, from first type Bessel functions $J_n(x)$:

$$e^{-r^2} I_n(r^2) = \int_0^\infty u e^{-u^2/2} J_n^2(xu) du$$

We also introduce the average Larmor radius $\rho_{cth\alpha}$, calculated from the thermal velocity:

$$\boxed{\rho_{cT\alpha} = \frac{v_{T\alpha}}{\omega_{c\alpha}}} \qquad (1.23)$$

The expression for the integration of $f_{1\alpha}$ is :

$$\int_0^\infty du_{0\perp} u_{0\perp} \int_0^{2\pi} d\psi f_{1\alpha} = -\left[1 + (\omega - k_y v_{d\alpha}) \sum_{n=-\infty}^{\infty} \frac{e^{-k_\perp^2 \rho_{cT\alpha}^2} I_n\left(k_\perp^2 \rho_{cT\alpha}^2\right)}{n\omega_{c\alpha} + k_y v_{f\alpha} + k_z v_{0z} - \omega} \right] \frac{n_{0\alpha}(x) q_\alpha}{k_B T_\alpha} \frac{e^{\frac{-v_z^2}{2 v_{T\alpha}^2}}}{\sqrt{2\pi} v_{T\alpha}} \hat{\varphi}_1$$

We further simplify the expression by introducing the function $\Lambda_n(r)$ pour $r \geq 0$:

$$\boxed{\Lambda_n(x^2) = e^{-x^2} I_n(x^2)} \qquad (1.24)$$

the integral is :

$$\int_0^\infty du_{0\perp} u_{0\perp} \int_0^{2\pi} d\psi_0 f_{1\alpha} =$$

$$-\left[1 + (\omega - k_y v_{d\alpha}) \sum_{n=-\infty}^{\infty} \frac{\Lambda_n\left(k_\perp^2 \rho_{cT\alpha}^2\right)}{n\omega_{c\alpha} + k_y v_{f\alpha} + k_z v_{0z} - \omega} \right] \frac{n_{0\alpha}(x) q_\alpha}{k_B T_\alpha} \frac{e^{\frac{-v_z^2}{2 v_{T\alpha}^2}}}{\sqrt{2\pi} v_{T\alpha}} \hat{\varphi}_1$$

Parallel velocity v_{0z} and wave-particle resonance

The integration on v_{0z} is improper: the expression shows a real pole by the denominator $n\omega_{c\alpha} + k_y v_{f\alpha} + k_z v_z - \omega\tau$. This denominator is equal to zero for particles whose velocity along the magnetic field is in resonance with the wave.

The integration on v_{0z} is proper, if the real axis is replaced by the contour Γ which follows the real axis $[-\infty,\infty]$ except near the pole :

$$v_{0z}=\frac{1}{k_z}\left(\omega\tau-n\omega_{c\alpha}-k_y v_{f\alpha}\right) .$$

The contour goes around the pole from below (by the negative imaginary if the pole is real): this makes it possible to guarantee causality. This prescription, known as Landau prscription, was the result of a long debate.

The 1st order density fluctuation is expressed:

$$n_{1\alpha}=-\frac{n_{0\alpha}(x)q_\alpha}{k_B T_\alpha}\hat{\varphi}_1\left[1+\frac{\omega-k_y v_{d\alpha}}{\sqrt{2\pi}v_{th\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)\int_\Gamma dv_{0z}\frac{e^{\frac{-v_{0z}^2}{2v_{T\alpha}^2}}}{n\omega_{c\alpha}-k_y v_{f\alpha}-k_z v_{0z}-\omega}\right]$$

In order to simplify the first term of the bracket, we used the property :

$$\int_{-\infty}^{+\infty}dv_{0z}\frac{1}{\sqrt{2\pi}v_{T\alpha}}e^{\frac{-v_{0z}^2}{2v_{T\alpha}^2}}=1$$

Plasma dispersion function

The integration along the direction parallel to the magnetic field introduces the plasma dispersion function $Z(z)$, as defined by Fried & Conte:

$$\boxed{Z(z)=\frac{1}{\sqrt{\pi}}\int_{\Gamma^-}dx\frac{e^{-x^2}}{x-z}} \qquad (1.25)$$

For $\Im(z)>0$, the contour can be the real axis:

$$Z(z)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dx\frac{e^{-x^2}}{x-z}$$

For $\Im(z)<0$, the value is deduced by the residue theorem:

$$Z(z)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dx\frac{e^{-x^2}}{x-z}+2i\sqrt{\pi}e^{-z^2}$$

For real z , the integration involves the Cauchy Principal Value:

$$Z(z)=\frac{1}{\sqrt{\pi}}PV\int_{-\infty}^{+\infty}dx\frac{e^{-x^2}}{x-z}+i\sqrt{\pi}e^{-z^2}$$

Another form of plasma dispersion function, $W(z)$, is also used:

$$W(z)=\frac{1}{\sqrt{2\pi}}\int_{\Gamma^-}dx\frac{x}{x-z}e^{-x^2/2}$$

Both functions correspond by the relation:

$$zZ(z)=W(\sqrt{2}\,z)-1$$

This function can also be related to other functions, such as the Faddeeva (or Faddeyeva) function $w(z)$:

$$Z(z)=i\sqrt{\pi}\,w(z)$$

This is a modified form of the complex complementary error function $erfc(z)$:

$$w(z)=e^{-z^2}erfc(-iz)$$

We use the Fried & Conte definition $Z(z)$ with the result parameter

$$z=\frac{\omega-n\omega_{c\alpha}-k_y v_{f\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}$$

and integration parameter

$$x=\frac{v_{0z}}{\sqrt{2}v_{T\alpha}}$$

giving the expression:

$$Z\left(\frac{\omega-n\omega_{c\alpha}+k_y v_{f\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)=\frac{|k_z|}{\sqrt{\pi}}\int_{\Gamma^-}dv_{0z}\frac{e^{-\frac{v_{0z}^2}{2v_{T\alpha}^2}}}{n\omega_{c\alpha}+k_y v_{f\alpha}+k_z v_{0z}-\omega}$$

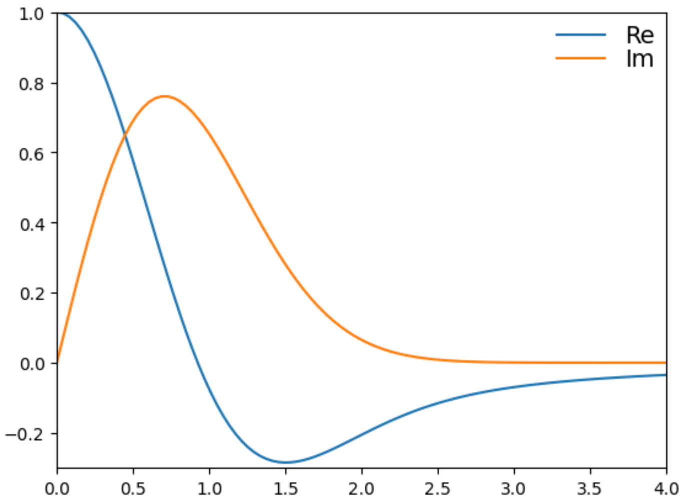


Figure 1: Plasma dispersion function

The density at the 1st order expression is:

$$\boxed{\hat{n}_{1\alpha}=-\frac{n_{0\alpha}(x)q_\alpha}{k_B T_\alpha}\hat{\varphi}_1\left[1+\frac{\omega-k_y v_{d\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)Z\left(\frac{\omega-n\omega_{c\alpha}-k_y v_{f\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right]} \quad (1.26)$$

2.e Mode dispersion relation

Calculation of the dielectric function

The dielectric function is defined by the sum of the susceptibilities $\chi_{\parallel\alpha}$:

$$\epsilon(\vec{k},\omega)=1-\sum_\alpha \chi_{\parallel\alpha}$$

with :

$$\chi_{\parallel\alpha}=\frac{q_\alpha \hat{n}_{1\alpha}}{\varepsilon_0 k^2\hat{\varphi}_1}$$

Susceptibility is expressed using the 1st order density expression $n_{1\alpha}$:

$$\chi_{\parallel\alpha}=-\frac{n_{0\alpha}(x)q_\alpha^2}{\varepsilon_0 k^2 k_B T_\alpha}\left[1+\frac{\omega-k_y v_{d\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)Z\left(\frac{\omega-n\omega_{c\alpha}-k_y v_{f\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right]$$

The expression shows the Debye length for the species α :

$$\boxed{\lambda_{D\alpha}(x)=\sqrt{\frac{\varepsilon_0 k_B T_\alpha}{n_{0\alpha}(x)\, q_\alpha^2}}} \tag{1.27}$$

The species susceptibility is expressed more directly :

$$\chi_{\|\alpha}=\frac{-1}{k^2\lambda_{D\alpha}^2}\left[1+\frac{\omega-k_y v_{d\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)Z\left(\frac{\omega-n\omega_{c\alpha}-k_y v_{f\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right]$$

Relation de dispersion

Poisson's equation is equivalent to canceling the dielectric function $\epsilon(\vec{k},\omega)=0$:

$$1-\sum_\alpha \chi_{\|\alpha}=0$$

The equation then establishes the dispersion relation of the mode :

$$\boxed{1+\sum_\alpha \frac{1}{k^2\lambda_{D\alpha}^2}\left[1+\frac{\omega-k_y v_{d\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)Z\left(\frac{\omega-n\omega_{c\alpha}-k_y v_{f\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right]=0} \tag{1.28}$$

2.f Phase between species density and potential

The expression for susceptibilities $\chi_{\|\alpha}$ shows that they can have a non-zero imaginary part. As the susceptibility is the ratio between the density of the species and the potential, up to real constants, this physically corresponds to a phase shift between the density and the potential.

If the argument of $\chi_{\|\alpha}$ is positive :

$$arg(\chi_{\|\alpha})>0$$

The potential and the density complex amplitudes argument follow this order :

$$arg(\hat{n_{1\alpha}})>arg(\hat{\varphi}_1)$$

Taking the temporal expression of these quantities :

$$\varphi_1(\vec{r},t)=\hat{\varphi}_1 e^{i(k_y y+k_z z-\omega t)} \quad ,$$

$$n_\alpha(\vec{r},t)=\hat{n}_{1\alpha} e^{i(k_y y+k_z z-\omega t)}$$

At the same position $\vec{r}$, the mode potential $\varphi_1(\vec{r},t)$ will be ahead of density $n_{1\alpha}(\vec{r},t)$.

1.3 Application for a drift wave

3.a Equation for a drift wave in a toric plasma

Plasma is limited to 2 species: electrons e^- and argon ions Ar^+ . The dielectric function is the sum :

$$\epsilon(\vec{k},\omega)=1-\chi_{\|e}(\vec{k},\omega)-\chi_{\|i}(\vec{k},\omega)$$

The drift velocity $\vec{v}_{f\alpha}$ is the sum of the particle drifts :

$$\vec{v}_{f\alpha}=\vec{v}_{E0}+\vec{v}_{B\alpha}$$

$\vec{v}_{E0}$ is the electric field drift, independent on the species:

$$\vec{v}_{E0} = \frac{-E_{0x}}{B_z}\vec{e}_y$$

$\vec{v}_{B\alpha}$ is the magnetic field curvature and gradient drift; the averaged value is

$$\vec{v}_{B\alpha} = -\frac{3}{2}\frac{m_\alpha v_{T\alpha}^2}{R_c q_\alpha B_z}\vec{e}_y$$

The diamagnetic drift $\vec{v}_{n\alpha}$ is:

$$\vec{v}_{n\alpha} = d_X \ln(n_{0\alpha})\frac{k_B T_\alpha}{q_\alpha B_z}\vec{e}_y$$

The drift velocity $\vec{v}_{d\alpha}$ is the sum of the all drifts :

$$\vec{v}_{d\alpha} = \vec{v}_{E0} + \vec{v}_{B\alpha} + \vec{v}_{n\alpha}$$

The mode wave vector $\vec{k}$ have no component along $\vec{e}_x$:

$$k_\perp = k_y$$

The Poisson equation is:

$$1 + \sum_{\alpha=i,e}\frac{1}{k^2\lambda_{D\alpha}^2}\left[1 + \frac{\omega - k_y(v_{E0}+v_{B\alpha}+v_{n\alpha})}{\sqrt{2}|k_z|v_{T\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)Z\left(\frac{\omega - n\omega_{c\alpha} - k_y(v_{E0}+v_{B\alpha})}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right] = 0$$

Effect of large scale $\vec{E}_0 \wedge \vec{B}_z$ drift

If we substitute ω with $\hat{\omega}$:

$$\hat{\omega} = \omega - k_y v_{E0}$$

the equation is the same with no $\vec{E}_0 \wedge \vec{B}_z$ drift:

$$1 + \sum_{\alpha=i,e}\frac{1}{k^2\lambda_{D\alpha}^2}\left[1 + \frac{\hat{\omega} - k_y(v_{B\alpha}+v_{n\alpha})}{\sqrt{2}|k_z|v_{T\alpha}}\sum_{n=-\infty}^{\infty}\Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right)Z\left(\frac{\hat{\omega} - n\omega_{c\alpha} - k_y v_{B\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right] = 0$$

Since the $\vec{E}_0 \wedge \vec{B}_z$ drift is independent with the particles, it only has a Doppler effect on the real frequency.

The $\vec{E}_0 \wedge \vec{B}_z$ drift is next considered equal to 0.

The effect is different for $\vec{v}_{B\alpha}$ since this drift varies for different species.

Mode wavelength and Larmor radius

The effect of perpendicular velocities depends on the comparison between the average Larmor radius, $\rho_{cT\alpha}$, and the perturbation wavelength in the perpendicular plan:

$$\lambda_\perp = \frac{2\pi}{k_\perp}$$

Assuming the Larmor radius is small compared to the perturbation wavelength, $\rho_{cT\alpha} \ll \lambda_\perp$, non-zero order harmonics are neglected :

$$\forall n \neq 0,\ \Lambda_n\left(k_\perp^2\rho_{cT\alpha}^2\right) = 0$$

The Poisson equation is written:

$$1+\sum_{\alpha=i,e}\frac{1}{k^2\lambda_{D\alpha}^2}\left[1+\frac{\hat{\omega}-k_y(v_{B\alpha}+v_{n\alpha})}{\sqrt{2}|k_z|v_{T\alpha}}\Lambda_0(k_\perp^2\rho_{cT\alpha}^2)Z\left(\frac{\hat{\omega}-k_y v_{B\alpha}}{\sqrt{2}|k_z|v_{T\alpha}}\right)\right]=0$$

3.b Approximated solution for purely drift waves

We next neglect the magnetic field curvature and gradient drift:

$$\vec{v}_{B\alpha}=0$$

Negligible electron radius

The value of Λ_0 at 0 corresponds to cases for which the Larmor radius is negligeable

$$\Lambda_0(0)=1$$

In the limit of the very small electron Larmor radius compared to the mode wavelength, the effect of the Larmor radius will be completely neglected:

$$\Lambda_0(k_y^2\rho_{cTe}^2)=1$$

The electron susceptibility is simplified:

$$\chi_{\parallel e}=\frac{-1}{k^2\lambda_{De}^2}\left[1+\frac{\omega-k_y(v_{Be}+v_{ne})}{\sqrt{2}|k_z|v_{Te}}Z\left(\frac{\omega-k_y v_{Be}}{\sqrt{2}|k_z|v_{Te}}\right)\right]$$

Electron plasma dispersion function approximation

The electron thermal velocity is large compared to the wave parallel phase velocity:

$$\omega-k_y v_{fe}\ll\sqrt{2}|k_z|v_{Te}$$

the electrons marginally resonate with the wave. The plasma dispersion function for electrons is limited to values near 0:

$$Z(z)\underset{z\to 0}{\sim} i\sqrt{\pi}$$

The electron parallel susceptibility is simplified :

$$\chi_{\parallel e}=\frac{-1}{k^2\lambda_{De}^2}\left[1+i\sqrt{\frac{\pi}{2}}\frac{\omega-k_y v_{ne}}{|k_z|v_{Te}}\right]$$

Ion plasma dispersion function approximation

If the wave parallel phase velocity is large compared to the particle thermal velocity, The plasma distribution function can be approximated by an asymptotic expansion for $z\to\infty$:

$$|\omega-n\omega_{c\alpha}-k_y v_{f\alpha}|\gg\sqrt{2}|k_z|v_{T\alpha}$$

This limit generally applies to ions, slower than the wave phase velocity.

$$Z(z)\underset{z\to\infty}{\sim}\frac{-1}{z}\left(1+\frac{1}{2z^2}\right)+i\sqrt{\pi}e^{-z^2}$$

For ions, as the ratio is less unbalanced, we will use a development limited to the 1st order:

$$Z(z)\sim\frac{-1}{z}+i\sqrt{\pi}e^{-z^2}$$

The ion parallel susceptibility expression is:

$$\chi_{\|i}=\frac{-1}{k^2\lambda_{Di}^2}\left[1+\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\left(-\frac{\omega-k_y v_{ni}}{\omega}+i\sqrt{\frac{\pi}{2}}\frac{\omega-k_y v_{ni}}{|k_z|v_{Ti}}e^{\frac{-\omega^2}{2k_z^2v_{Ti}^2}}\right)\right]$$

the dielectric function is written:

$$\boxed{\epsilon=1+\frac{1}{k^2\lambda_{De}^2}+\frac{1}{k^2\lambda_{Di}^2}\left[1-\frac{\omega-k_y v_{ni}}{\omega}\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\right]+i\sqrt{\frac{\pi}{2}}\left[\frac{\omega-k_y v_{ne}}{k^2\lambda_{De}^2|k_z|v_{Te}}+\frac{\Lambda_0\left(k_y^2\rho_{cTi}^2\right)}{k^2\lambda_{Di}^2}\frac{\omega-k_y v_{ni}}{|k_z|v_{Ti}}e^{\frac{-\omega^2}{2k_z^2v_{Ti}^2}}\right]}$$

Drift wave frequency

To extract the real frequency of the mode, we solve:

$$\Re[\epsilon(\vec{k},\omega_R)]=0$$

or :

$$1+\frac{1}{k^2\lambda_{De}^2}+\frac{1}{k^2\lambda_{Di}^2}\left[1-\frac{\omega_R-k_y v_{ni}}{\omega_R}\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\right]=0$$

Using the relationship between the ion and electron diamagnetic drifts:

$$v_{ni}=-\frac{T_i}{T_e}v_{ne}$$

The solution of the equation on ω_R is:

$$\omega_R=k_y v_{ne}\frac{\Lambda_0\left(k_y^2\rho_{cTi}^2\right)}{1+\frac{T_e}{T_i}\left(1-\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\right)+k^2\lambda_{De}^2}$$

We note that ω_R has the same sign as $k_y v_{ne}$: the drift wave propagates in the same direction as the electron diamagnetic drift, with a slightly slower phase velocity.

Drift wave growth rate

The growth rate is calculated by the formula :

$$\gamma=\frac{-\Im[\epsilon(\vec{k},\omega_R)]}{\partial_\omega\Re[\epsilon(\vec{k},\omega_R)]}$$

The real part derivative of the dielectric function gives:

$$\partial_\omega\Re[\epsilon(\vec{k},\omega_R)]=-\frac{\Lambda_0\left(k_y^2\rho_{cTi}^2\right)k_y v_{ni}}{k^2\lambda_{Di}^2\omega_R^2}$$

and the imaginary part of the dielectric function expression is:

$$\Im[\epsilon(\vec{k},\omega_R)]=\frac{1}{k^2\lambda_{De}^2}\sqrt{\frac{\pi}{2}}\frac{\omega_R-k_y v_{ne}}{|k_z|v_{Te}}+\frac{1}{k^2\lambda_{Di}^2}\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\sqrt{\frac{\pi}{2}}\frac{\omega_R-k_y v_{ni}}{|k_z|v_{Ti}}e^{\frac{-\omega_R^2}{2k_z^2v_{Ti}^2}}$$

The growth rate is written after rearrangement of the factors:

$$\gamma=\sqrt{\frac{\pi}{2}}\frac{\omega_R^2}{\Lambda_0\left(k_y^2\rho_{cTi}^2\right)}\left[\left(1-\frac{\omega_R}{k_y v_{ne}}\right)\frac{1}{|k_z|v_{Te}}-\left(1+\frac{T_e}{T_i}\frac{\omega_R}{k_y v_{ne}}\right)\frac{\Lambda_0\left(k_y^2\rho_{cTi}^2\right)}{|k_z|v_{Ti}}e^{\frac{-\omega_R^2}{2k_z^2v_{Ti}^2}}\right]$$

Since $0<\Lambda_0<1$:

$$0<\frac{\omega_R}{k_y v_{ne}}<1$$

the electron component of the growth rate (the first term in square brackets) is destabilizing, while the ion component (the second term) is stabilizing.

Because of the exponential $e^{-\omega_R^2/2k_z^2 v_{Ti}^2}$, the ion component is generally much lower in absolute value: the electronic component dominates, the ion component is neglected.

Using the expression of ω_R :

$$\gamma=\sqrt{\frac{\pi}{2}}\frac{k_y^2 v_{ne}^2}{|k_z| v_{Te}}\Lambda_0\left(k_\perp^2\rho_{cTi}^2\right)\frac{\left(1+\frac{T_e}{T_i}\right)\left(1-\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\right)+k^2\lambda_{De}^2}{\left[1+\frac{T_e}{T_i}\left(1-\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\right)+k^2\lambda_{De}^2\right]^3}$$

3.c Back to the fluid model

Small ion Larmor radius

To find the formula obtained by the fluid model, we consider the average ion Larmor radius is small compared to the perpendicular wavelength ($k_y^2\rho_{cTi}^2 \ll 1$) :

$$\Lambda_0\left(k_y^2\rho_{cTi}^2\right)\sim 1-k_y^2\rho_{cTi}^2$$

and neglecting the Debye length with respect to the mode wavelength:

$$k^2\lambda_{De}^2 \ll 1$$

the frequency is written:

$$\omega_R=k_y v_{ne}\frac{1-k_y^2\rho_{cTi}^2}{1+\frac{T_e}{T_i}k_y^2\rho_{cTi}^2}$$

we introduce ρ_{cths} is the Larmor radius evaluated with the ion mass and the electron velocity, as the ion acoustic wave velocity $c_s=\sqrt{k_B T_e/m_i}$:

$$\rho_{cs}=\frac{c_s}{\omega_{ci}}=\sqrt{\frac{T_e}{T_i}}\rho_{cTi}$$

The frequency is simplified:

$$\omega_R=k_y v_{ne}\frac{1}{1+k_y^2\rho_{cTs}^2}$$

The growth rate is simplified :

$$\gamma=\sqrt{\frac{\pi}{2}}\frac{k_y^2 v_{ne}^2}{|k_z| v_{Te}}\frac{k_y^2\left(\rho_{cTi}^2+\rho_{cs}^2\right)}{\left(1+k_y^2\rho_{cs}^2\right)^3}$$

Annex : The Landau Damping

Landau damping is a resonance effect between an electrostatic wave and the electrons. It has the effect of damping or amplifying the wave and distorting the electron velocity distribution.

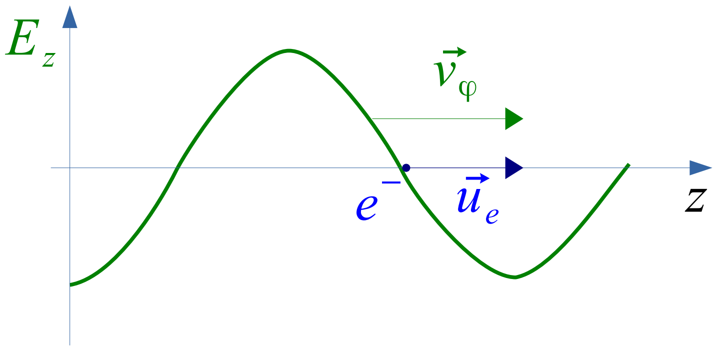
Landau damping is the most emblematic purely kinetic phenomenon in plasma physics: it is not accessible by fluid models.

It plays an important role in the development of electrostatic instabilities in plasmas.

The phenomenon was discovered by L. Landau in 1945 (J. of Phys. 1946). It is one of the most discussed phenomena in classical physics since.

Kinetic phenomenon: Landau damping

Landau damping describes the damping of an electrostatic wave by the plasma electrons.



For this damping, collisions are neglected: there is no dissipation.

The effect is described by the interaction of an electrostatic (longitudinal) wave with a density electron distribution n_{e0} .

The electric field is a Fourier mono-mode :

$$E_\parallel(z,t)=E_{\parallel 0}\cos(k_\parallel z-\omega t)$$

Since the force applied on the electron is oscillating, its effect will be on average zero, except for electrons whose longitudinal speed u_e is infinitely close to the wave phase velocity

$$v_\varphi=\omega/k_\parallel \ ,$$

through a wave-particle resonance effect.

The electrons that are very slightly slower than the wave are accelerated on average. Very slightly faster electrons will on average be slowed down.

The energy gain or loss over the entire distribution depends on the population difference between the slightly faster and the slightly slower electrons. The energy balance will depend on the derivative of the longitudinal velocity distribution of the electrons, $f_\parallel$ for the wave phase velocity.

Perturbative expansion

The following arguments are not a model, but an interpretation of the phenomenon.

The wave is a longitudinal electrostatic wave:

$$E_{\parallel}(z,t)=E_{\parallel 0}\cos(k_{\parallel}z-\omega t)$$

A charged particle moves in the same direction:

$$u_{\parallel}(0)=u_z$$

$$z(0)=z_i$$

The particle motion equation in this field is :

$$a_{\parallel}=\frac{-q_e}{m_e}E_{\parallel 0}\cos(k_{\parallel}z(t)-\omega t)$$

This equation has no analytical solution. We will assume the amplitude of the wave is sufficiently low to carry out a perturbative expansion.

$$u_{\parallel}=u_{\parallel 0}+u_{\parallel 1}+\ldots$$

$$z_{\parallel}=z_{\parallel 0}+z_{\parallel 1}+\ldots$$

Electron perturbed motion

The 0th order corresponds to the electron motion unperturbed by the wave

$$z_0(t)=z_i+u_{\parallel 0}t$$

$$u_z=u_{\parallel 0}$$

The acceleration at the 1st order is estimated from the motion at the 0th order:

$$a_{\parallel 1}=\frac{-q_e}{m_e}E_{\parallel 0}\cos(k_{\parallel}z_i+k_{\parallel}u_z t-\omega t)$$

The expression shows $\hat{\omega}$ the wave frequency observed in the reference frame of the unperturbed electron:

$$\hat{\omega}=\omega-k_{\parallel}u_z$$

The solution for the velocity term at the 1st order ($u_{\parallel 1}(0)=0$) :

$$u_{\parallel 1}=\frac{q_e}{m_e}E_{\parallel 0}\frac{\sin(k_{\parallel}z_i-\hat{\omega}t)-\sin(k_{\parallel}z_i)}{\hat{\omega}}$$

$$z_{\parallel 1}=\frac{q_e}{m_e}E_{\parallel 0}\left[\frac{\cos(k_{\parallel}z_i-\hat{\omega}t)-\cos(k_{\parallel}z_i)}{\hat{\omega}^2}-\frac{t\sin(k_{\parallel}z_i)}{\hat{\omega}^2}\right]$$

Energy transfer and velocity distribution

One particle energy transfer

The electron energy time derivative is equal to the electric force power on the electron:

$$\frac{d}{dt}E_c=-q_e u_{\parallel}E_{\parallel}$$

We use the expansion up to the perturbation 2nd order:

$$u_{\parallel}E_{\parallel}=(u_{\parallel 0}+u_{\parallel 1})E_{\parallel}(z_0+z_1,t)$$

$$u_{\parallel}E_{\parallel}=(u_{\parallel 0}+u_{\parallel 1})\left[E_{\parallel}(z_0,t)+\frac{\partial}{\partial z}E_{\parallel}(z_0,t)z_1\right]$$

$$\frac{d}{dt}E_c=-q_e\left[u_{\parallel 0}E_{\parallel}(z_0,t)+u_{\parallel 0}\frac{\partial}{\partial z}E_{\parallel}(z_0,t)z_1+u_{\parallel 1}E_{\parallel}(z_0,t)\right]$$

The energy transfer is estimated by averaging this effect over the electron velocity and space distribution.

The electron wave initial phase is random: we average over all the electron initial positions over a wavelength distance:

$$\frac{d}{dt}\bar{E}_c=\frac{k_{\parallel}}{2\pi}\int_0^{2\pi/k_{\parallel}}\frac{d}{dt}E_c dz_i$$

Linear terms in $E_{\parallel 0}$ are on average zero because of the oscillating factor. Only certain quadratic terms will remain, typically those of the form ($i=1{,}2$):

$$\sin(k_{\parallel}z_i-\hat{\omega}t)\sin(k_{\parallel}z_i)=\frac{1}{2}\left[\cos(\hat{\omega}t)+\cos(2k_{\parallel}z_i-\hat{\omega}t)\right]$$

The time derivative of the energy averaged over the initial phase simplifies to:

$$\frac{d}{dt}\bar{E}_c=\frac{q_e^2}{2m_e}E_{\parallel 0}^2\left[\frac{\omega}{\hat{\omega}^2}\sin(\hat{\omega}t)-\frac{\omega t}{\hat{\omega}}\cos(\hat{\omega}t)+t\cos(\hat{\omega}t)\right]$$

Distribution-averaged energy transfer

The energy time derivative is averaged over the initial phase:

$$\frac{d}{dt}\bar{E}_c=\frac{q_e^2}{2m_e}E_{\parallel 0}^2\left[\frac{\omega}{\hat{\omega}^2}\sin(\hat{\omega}t)-\frac{\omega t}{\hat{\omega}}\cos(\hat{\omega}t)+t\cos(\hat{\omega}t)\right]$$

The cases for which $\hat{\omega}$ is close 0 dominate: they correspond to a resonance effect between the wave and the electron. This average remains finite when $\hat{\omega}\to 0$.

We now average over the electron velocity distribution we assume to be homogeneous:

$$f_e(\vec{u}_{\perp},u_z)$$

The normalized distribution in the parallel direction is:

$$f_{\parallel}(u_z)=\frac{1}{n_{e0}}\iint f_e(\vec{u}_{\perp},u_z)d^2\vec{u}_{\perp}$$

The electron energy per unit volume is defined:

$$S_c=\iiint_{\vec{u}}\bar{E}_c f_e(\vec{u}_{\perp},u_z)d^2\vec{u}_{\perp}du_z$$

Its time derivative is then expressed:

$$\frac{d}{dt}S_c=\iiint_{\vec{u}}\frac{d}{dt}\bar{E}_c f_e(\vec{u}_{\perp},u_z)d^2\vec{u}_{\perp}du_z$$

or:

$$\frac{d}{dt}S_c=n_{e0}\int_{u_z}\frac{d}{dt}\bar{E}_c f_{\parallel}(u_z)du_z$$

$$\frac{d}{dt}S_c=n_{e0}\int_{u_z}\frac{q_e^2}{2m_e}E_{\parallel 0}^2\left[\frac{\omega}{\hat{\omega}^2}\sin(\hat{\omega}t)-\frac{\omega t}{\hat{\omega}}\cos(\hat{\omega}t)+t\cos(\hat{\omega}t)\right]f_{\parallel}(u_z)du_z$$

Since the factor in square brackets can be considered as a derivative:

$$\frac{\omega}{\hat{\omega}^2}\sin(\hat{\omega}t)-\frac{\omega t}{\hat{\omega}}\cos(\hat{\omega}t)+t\cos(\hat{\omega}t)=-\frac{\partial}{\partial\hat{\omega}}\left[\frac{\omega\sin(\hat{\omega}t)}{\hat{\omega}}-\sin(\hat{\omega}t)\right]$$

or since $\hat{\omega}=\omega-k_{\|}u_z$:

$$\frac{\omega}{\hat{\omega}^2}\sin(\hat{\omega}t)-\frac{\omega t}{\hat{\omega}}\cos(\hat{\omega}t)+t\cos(\hat{\omega}t)=\frac{1}{k_{\|}}\frac{\partial}{\partial u_z}\left[\frac{\omega\sin(\hat{\omega}t)}{\hat{\omega}}-\sin(\hat{\omega}t)\right]$$

The expression can be integrated by parts:

$$\frac{d}{dt}S_c=-\frac{q_e^2 n_{e0}E_{\|0}^2}{2m_e k_{\|}}\int_{u_{\|}}\left[\frac{\omega\sin(\hat{\omega}t)}{\hat{\omega}}-\sin(\hat{\omega}t)\right]f'_{\|}(u_z)\,du_z$$

As $t\rightarrow\infty$, for frequencies $\hat{\omega}\gg 1/t$, the integration will decrease rapidly because of the oscillating factor: the resonant frequency $\hat{\omega}\sim 0$ will dominate: . This corresponds to:

$$u_z\sim\omega/k_{\|}\ .$$

For $\hat{\omega}\sim 0$, the velocity distribution varies slowly around the resonance. We consider :

$$f_{\|}(u_z)\sim_{\hat{\omega}\sim 0}f_{\|}(\omega/k_{\|})$$

In the brackets $\left[\frac{\omega\sin(\hat{\omega}t)}{\hat{\omega}}-\sin(\hat{\omega}t)\right]$, the 1st term dominates over the 2nd:

$$\frac{d}{dt}S_c=-\frac{q_e^2 n_{e0}E_{\|0}^2\omega}{2m_e k_{\|}^2}f'_{\|}\left(\frac{\omega}{k_{\|}}\right)\int\frac{\sin(x)}{x}dx$$

with the variable substitution:

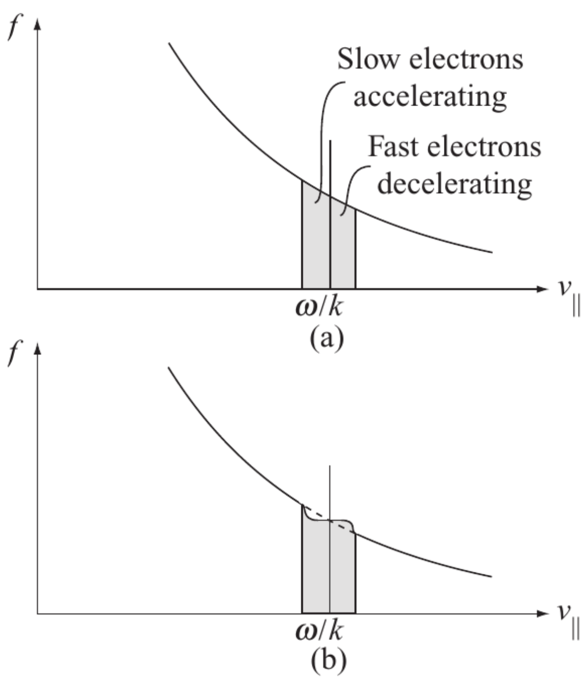
$$x=-\hat{\omega}t=(k_{\|}u_z-\omega)t$$

The electron energy gain per unit volume time derivative becomes :

$$\frac{d}{dt}S_c=-\frac{\pi q_e^2 n_{e0}E_{\|0}^2\omega}{2m_e k_{\|}{}^2}f'_{\|}\left(\frac{\omega}{k_{\|}}\right)$$

For times longer than the wave frequency, the energy transfer becomes independent of time.

Energy and and distribution derivative



The energy transfer sign depends on the electron parallel velocity distribution derivative, for the wave phase velocity.

For a Maxwellian distribution :

$$f_{\parallel}(u_{\parallel})=\sqrt{\frac{m_e}{2\pi k_B T_e}}\, e^{\frac{-m_e u_{\parallel}^2}{2 k_B T_e}}$$

Its derivative is:

$$f'_{\parallel}(u_{\parallel})=\frac{-u_{\parallel}}{\sqrt{2\pi}}\left(\frac{m_e}{k_B T_e}\right)^{\frac{3}{2}} e^{\frac{-m_e u_{\parallel}^2}{2 k_B T_e}}$$

The derivative is negative everywhere: the electrons gain energy; the wave is damped.

When the velocity distribution is not Maxwellian (or simply decentered because of a drift velocity) and presents velocity range where the velocity distribution derivative is positive, the electrons will transmit their energy to the wave: electrostatic instability may appear spontaneously.

The interaction between the wave and the distribution of electrons has the effect of distorting the velocity distribution of the electron.

References

W. M. Stacey, Fusion plasma physics, Wiley-vch Verlag GmbH & Co. (2005)
Chap. 8.6 : Drift wave instabilities

F. F. Chen, Plasma physics and controlled fusion, 2nd ed., Plenum Press (1984)
Chap. 6 : Equilibrium and stability

D. G. Swanson, Plasma waves, Academic Press Inc. (1989)
Chap. 6.5 : Drift waves and instabilities

S. Brunner, Waves and instabilities in inhomgeneous plasmas, EPFL Lecture (2003)