Instabilities Observations in magnetized plasmas : TORIX with probes

LPP
Laboratoire de Physique des Plasmas

Practical Work 27/02/26

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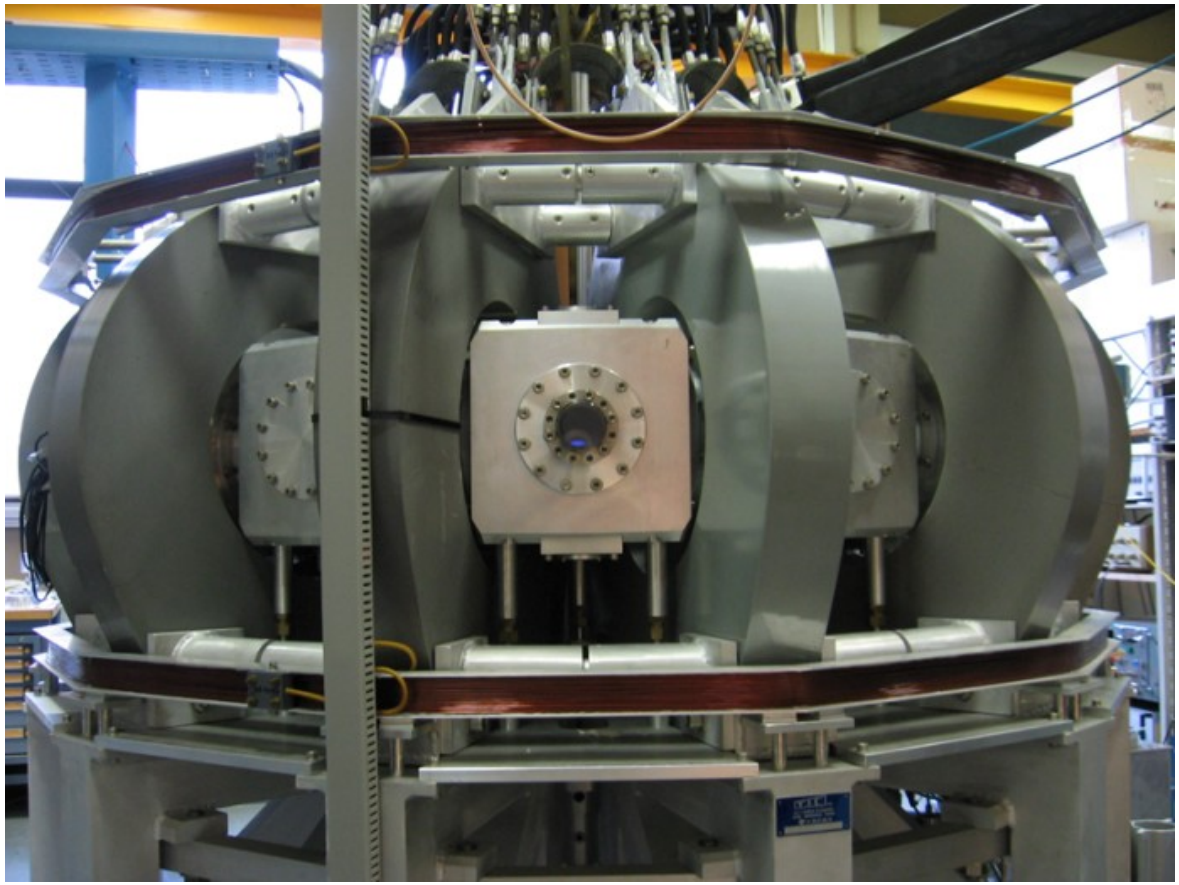




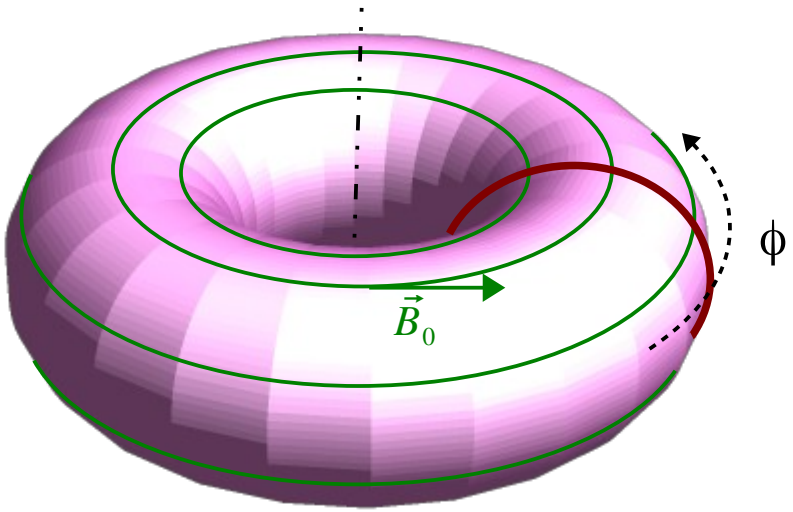


Instabilities in magnetized plasmas

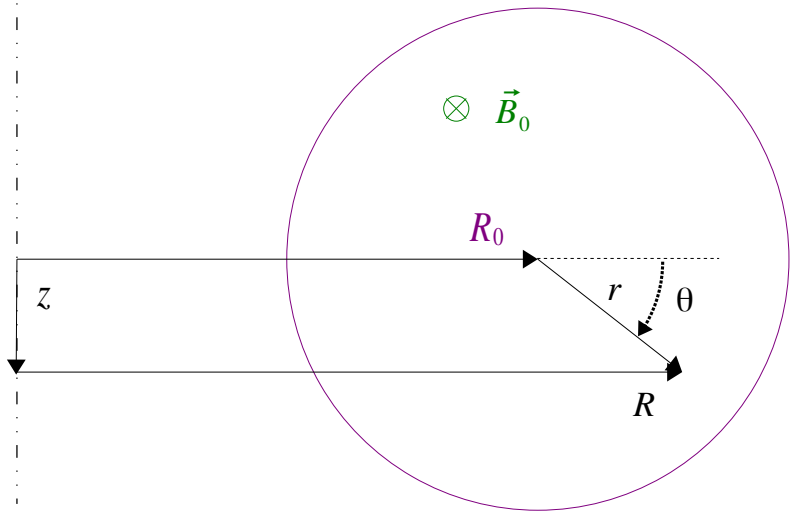
- Torix plasma "equilibrium"
- Drift wave instabilities
- Langmuir Probe Instability measurements



Toric geometry



Poloidal section



We use usual toric geometry coordinate system

ϕ : toroidal angle

Inside a poloidal section

R_0 : tore large radius

R : particle large radius

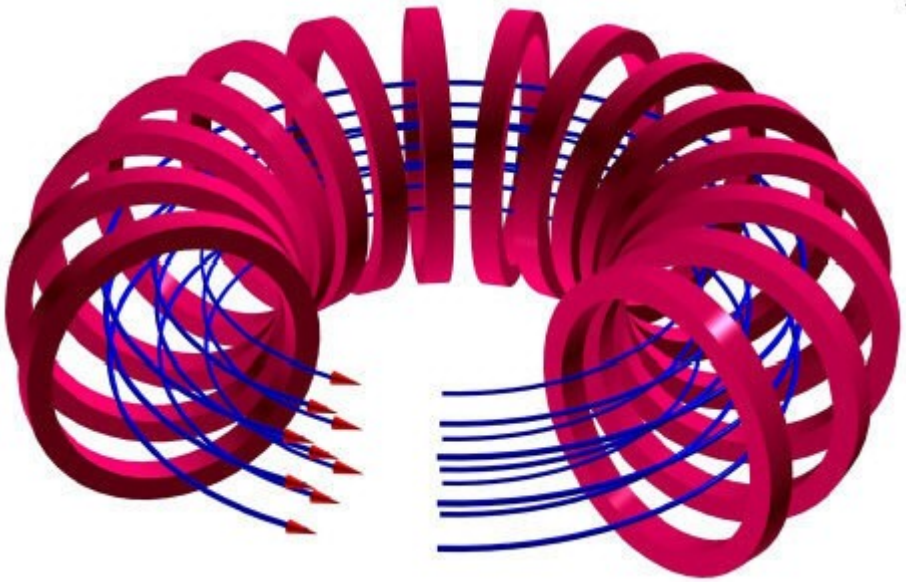
z : vertical coordinate

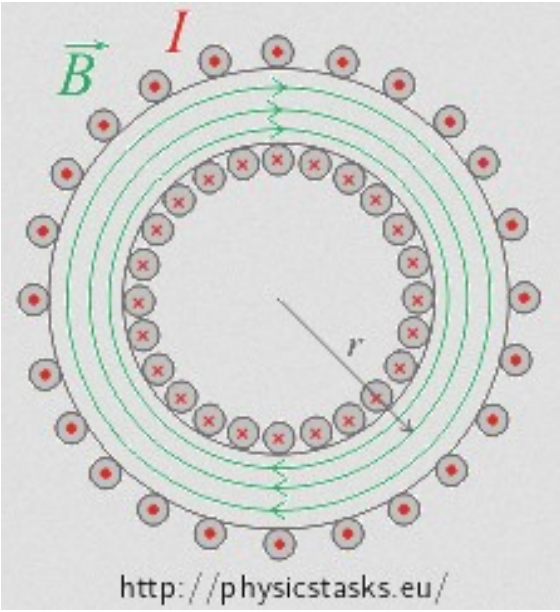
r : particle small radius

θ : poloidal angle

Magnetic field geometry

$$\oint_\Gamma \vec{B}_0 . \vec{dl} = \mu_0 I_b$$





The main magnetic is created by coils around the tore device
The magnetic field lines are circular.
The magnetic field magnitude decreases with the large radius:

$$B = \frac{\mu_0}{2\pi R} I_b$$

Main drifts : E x B drift

Plasma with a main magnetic field $\vec{B}_0$
and an uniform electric field $\vec{E}$

$$m_\alpha \vec{a}_\alpha = q_\alpha (\vec{E} + \vec{u}_\alpha \wedge \vec{B}_0)$$

Particle cyclotron motion is modified :

$$x_\alpha(t) = x_{g\alpha} + \frac{u_{\perp\alpha}}{\omega_{c\alpha}} \sin[\omega_{c\alpha} t + \psi]$$

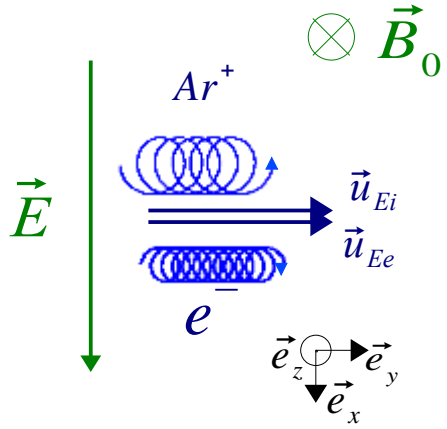
$$y_\alpha(t) = y_{g\alpha 0} - \frac{E}{B_0} t + \frac{u_{\perp\alpha}}{\omega_{c\alpha}} \cos[\omega_{c\alpha} t + \psi]$$

The drift velocity is perpendicular to $\vec{B}_0$ and $\vec{E}$:

$$\vec{u}_{E\alpha} = \frac{\vec{E} \wedge \vec{B}_0}{B_0^2}$$

This drift does not depend on the particle charge:

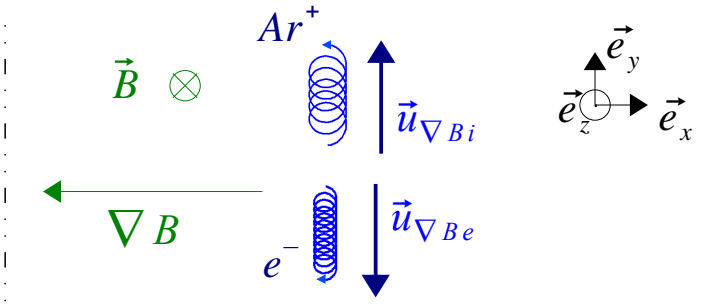
The whole plasma drifts.



Magnetic field gradient and curvature drifts

In presence of magnetic field magnitude gradient $\vec{\nabla} B$ there is also a particle drift :

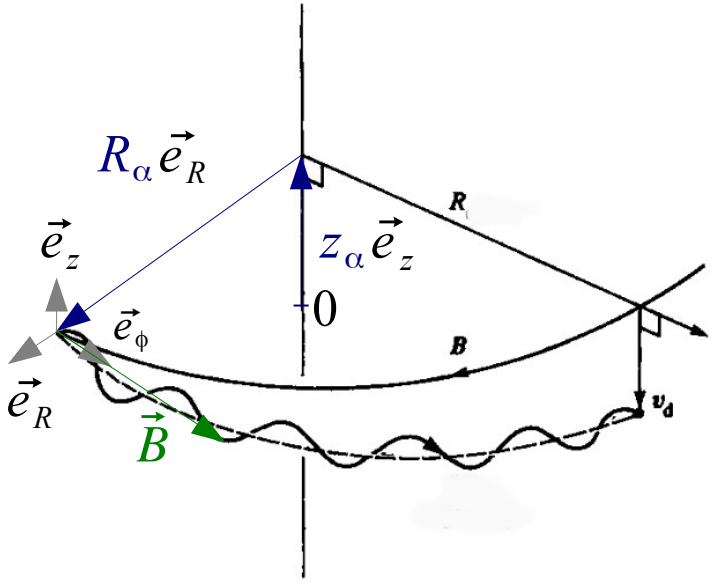
$$\vec{u}_{g\perp\alpha} = \frac{u_{\perp\alpha}^2}{2\omega_{c\alpha}} \frac{\vec{B}\wedge\vec{\nabla} B}{B^2}$$



R

In presence of magnetic field line curvature $\frac{1}{R_B}$ there is also a particle drift :

$$\vec{u}_{g\perp\alpha} = \frac{u_{\parallel\alpha}^2}{\omega_{c\alpha}} \frac{\vec{e}_R\wedge\vec{B}}{R_B B}$$



For a toric magnetic field both drifts are present : $B = \frac{B_0 R_0}{R}$ $\quad R_B = R$

$$\vec{u}_{torB\alpha} = \frac{1}{\omega_{c\alpha}}\left(u_{\parallel\alpha}^2 + \frac{u_{\perp\alpha}^2}{2}\right)\frac{\vec{e}_z}{R}$$

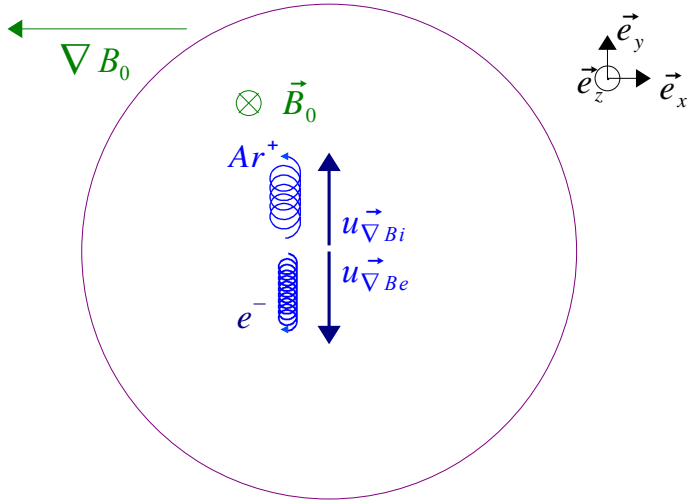
Toroidal magnetic field drifts

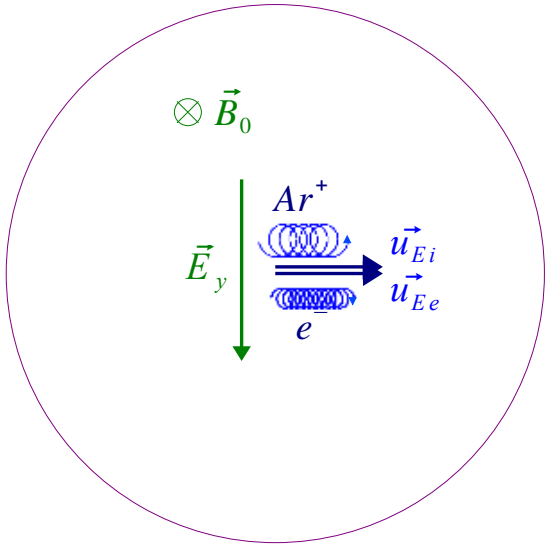
For a toric plasma, because of the magnetic field gradient and curvature, charged particles undergo vertical drifts, with opposite directions for opposite charge signs.

This create a vertical charge separation that generates a secondary vertical electric field.

This vertical electrical field induces a plasma radially outward ExB drift:

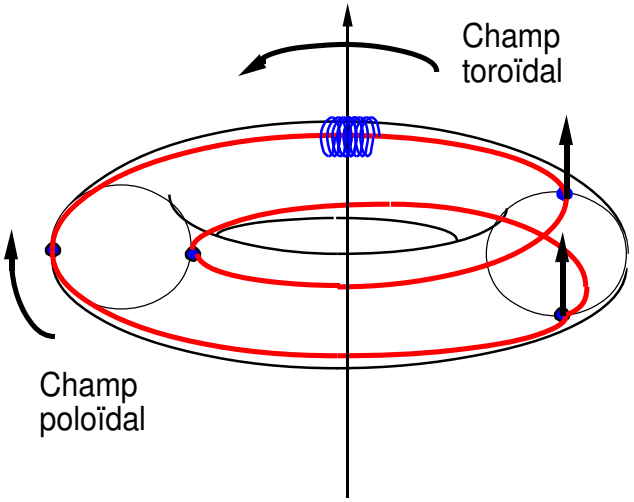
$$\vec{u}_{E\alpha} = \frac{\vec{E}_y \wedge \vec{B}_0}{B_0^2}$$





Helical magnetic field lines

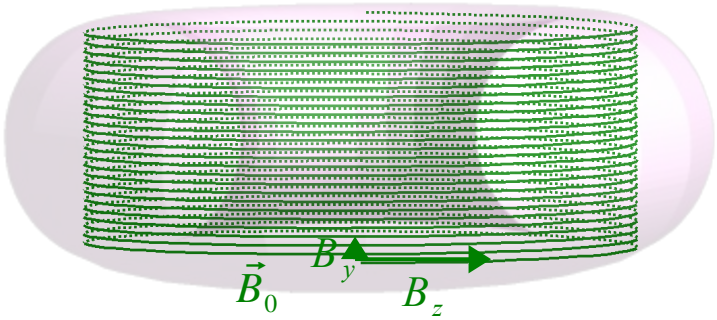
Stellarator or Tokamak :
Poloidal helicoid magnetic field line geometry is necessary to compensate the initial magnetic field gradient and curvature drifts.



For Torix an additional magnetic field component is added:

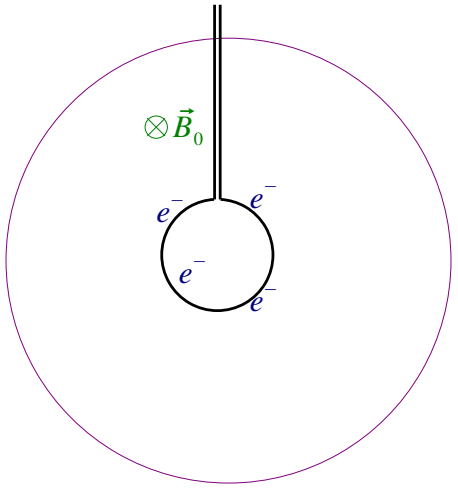
- since charged particles are free to move along the magnetic field lines, they can **compensate part of the charge separation**: the secondary drift is reduced;
- the magnetic field lines are then open; **charged particles can move to the edge**.

For small vertical magnetic field component, the 1st effect is the main one.

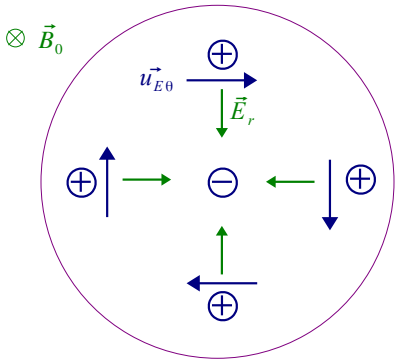


Polarized filament

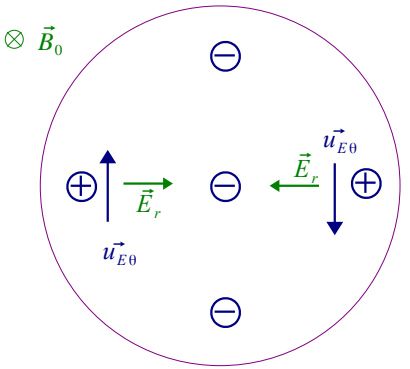
Plasma is created by a hot polarized (negative potential -90 V) tugnsten filament that emits primary electron inside Argon gas. The energetic primary electrons ionize part of the argon atoms.



The negative potential in the center of the tore due to the filament implies a circular plasma E x B drift around the minimum potential



When the a additional vertical magnetic field component is present, the potential is more uniform vertically. E x B drifts are mainly vertical.

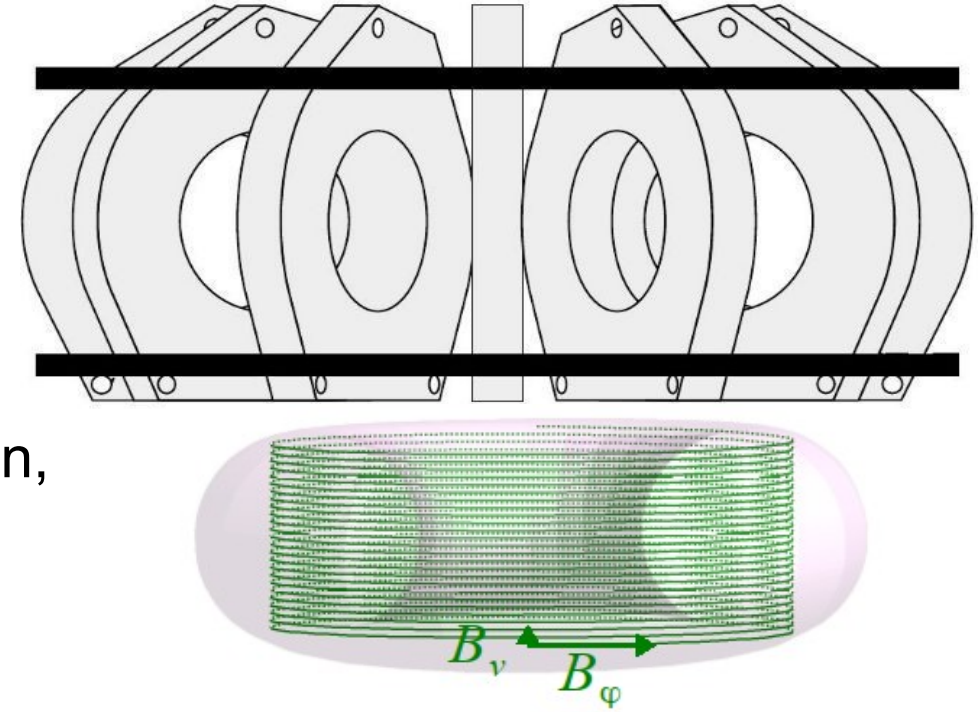


Torix parameters

Torix : toric magnetized plasma

Large radius : $R = 61\,cm$

Small radius : $a = 10\,cm$



- 12 coils around the tore to create the main toroidal magnetic field $I \sim 1\,kA$
- 2 horizontal coils in Helmholtz configuration, to add a small vertical magnetic field component

$B_\varphi \leq 0.4\,T$ $B_v \leq 2\,mT$

Cold magnetized plasma typical parameters :

Neutral gas : Argon $n_{Ar} \sim 10^{19}\,m^{-3}$

Ionization : $n_e \sim 10^{17}\,m^{-3}$

Electron heating : $k_B T_e \sim eV$

Ion temperature : $k_B T_{Ar+} = .03\,eV$

Drift wave instabilities

- Diamagnetic drift velocity
- Drift wave main

Diamagnetic drift : fluid approach

Charged particle velocity distribution with a density or a temperature gradient

For any species α (electron or ion) $P_\alpha = \gamma_\alpha n_\alpha k_B T_\alpha$

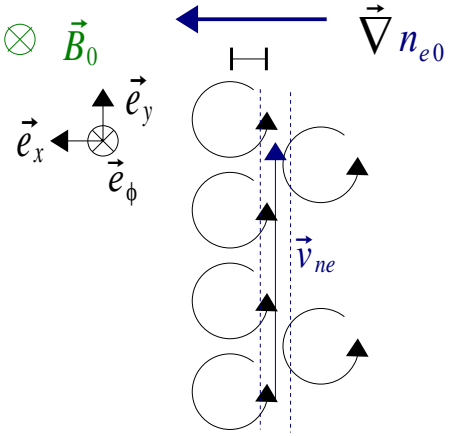
$\vec{\nabla} n_{\alpha 0}$ $\vec{\nabla} T_e$

Momentum conservation equation :

$$n_\alpha m_\alpha \left(\frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha . \nabla \vec{u}_\alpha \right) = n_\alpha q_\alpha \vec{u}_\alpha \times \vec{B}_0 - \vec{\nabla} P_\alpha$$

Stationary state : $n_\alpha q_\alpha \vec{u}_\alpha \times \vec{B}_0 - \vec{\nabla} P_\alpha = \vec{0}$

Parallel to $\vec{B}_0$ component : $\vec{u_{\alpha\|}} = \vec{0}$

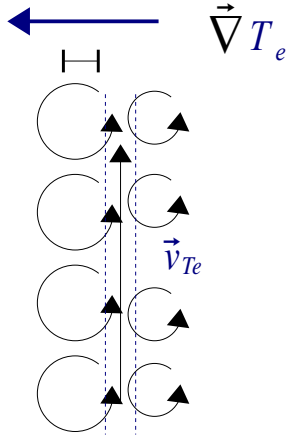


Perpendicular velocity part :

$$\vec{u}_{P\alpha} = \frac{-1}{n_\alpha q_\alpha B_0^2} \vec{\nabla} P_\alpha \times \vec{B}_0$$

$\vec{B}_0 \perp \vec{\nabla} P_\alpha$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} . \vec{c}) \vec{b} - (\vec{b} . \vec{c}) \vec{a}$



$$\vec{u}_{P\alpha} = \frac{-\gamma_\alpha k_B T_\alpha}{q_\alpha B_0} \frac{\vec{\nabla} P_\alpha}{P_\alpha} \times \frac{\vec{B}_0}{B_0}$$

$$\vec{u}_{P\alpha} = -u_{T\alpha} \rho_{cT\alpha} \frac{\vec{\nabla} P_\alpha}{P_\alpha} \times \frac{\vec{B}_0}{B_0}$$

$$u_{T\alpha} = \sqrt{\frac{\gamma_\alpha k_B T_\alpha}{m_\alpha}} \qquad \rho_{cT\alpha} = \frac{u_{T\alpha}}{\omega_{c\alpha}} \qquad \omega_{c\alpha} = \frac{q_\alpha B_0}{m_\alpha}$$

Gradient length

Dimension :

$$\frac{P_\alpha}{|\nabla P_\alpha|} : [L]$$

Gradient length :

$$l_{P\alpha} = \frac{P_\alpha}{|\nabla P_\alpha|}$$

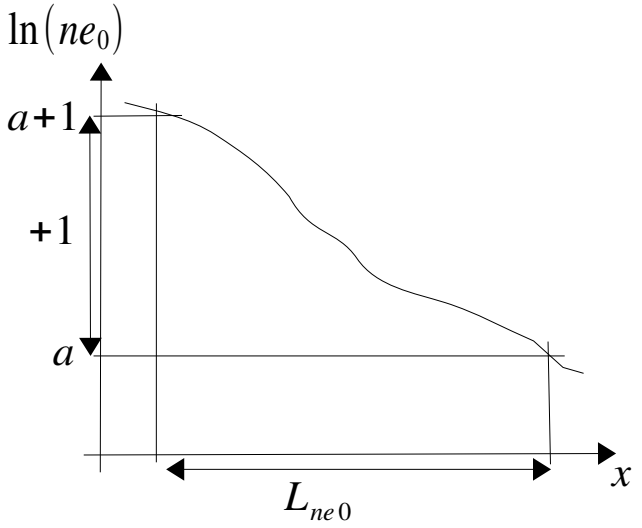
or :

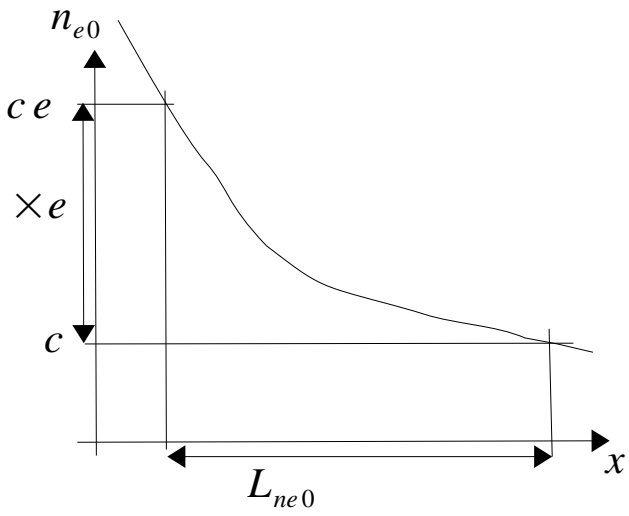
$$l_{P\alpha} = \frac{1}{|\nabla \ln P_\alpha|}$$

Length for which

$\ln P_\alpha$ varies with addition $+1$

or P_α varies of a ratio e.





Linear perturbation development

Plasma : steady state 0th order + perturbation 1st order

Steady state : $\vec{B}_0 \quad n_{e0}(x)\ \vec{\nabla} n_{e0} \quad T_{e0} \quad \varphi_0$

Perturbation : $n_{e1}(\vec{r},t) \quad \varphi_1(\vec{r},t)$

$$n_e(\vec{r},t)=n_{e0}(x)+n_{e1}(\vec{r},t)$$

The perturbation is small compared to the steady state :

$$\sqrt{\langle n_{e1}(\vec{r},t)^2\rangle_t} \ll n_{e0}(\vec{r})$$

We only consider a Fourier mode $n_{e1}(\vec{r},t)=\tilde{n}_{e1}\, e^{i(k_y y-\omega t)}$

1st order gradients and time derivatives have linear expressions :

$$\vec{\nabla} n_{e1}=i\,k_y\, n_{e1}\,\vec{e}_y \qquad \frac{\partial}{\partial t} n_{e1}=-i\,\omega\, n_{e1}$$

The goal is to extract a a dispersion relation (between k_y and ω).

If this relation is real, it only describes the mode propagation $\omega_R=f(k_y)$

If this relation has an imaginary part, it describes also the mode growth :

$\omega=\omega_R+i\,\gamma$ $\qquad$ γ : growth rate

$$n_{e1}(\vec{r},t)=\tilde{n}_{e1}\, e^{\gamma t}\, e^{i(k_y y-\omega_R t)}$$

$$\left|n_{e1}(\vec{r},t)\right|=\tilde{n}_{e1}\, e^{\gamma t}$$

Drift wave : mode propagation

Electron are supposed to be adiabatic:

$$n_{e0}+n_{e1}=n_{e0}\,e^{q_e\varphi_1/k_BT_e}$$

For small perturbations: $n_{e1}=\frac{q_e\varphi_1}{k_BT_e}n_{e0}$

Plasma potential perturbations are directly proportional to density perturbations (electron equation):

$$\varphi_1=\frac{k_BT_e}{q_en_{e0}}n_{e1}$$

Ions are following the perturbation ExB drift: $u_{ix1}=-i\,k_y\frac{\varphi_1}{B_{z0}}$

Ion mass conservation: $-i\,\omega\,n_{e1}+u_{ix1}\,\partial_x n_{e0}=0$

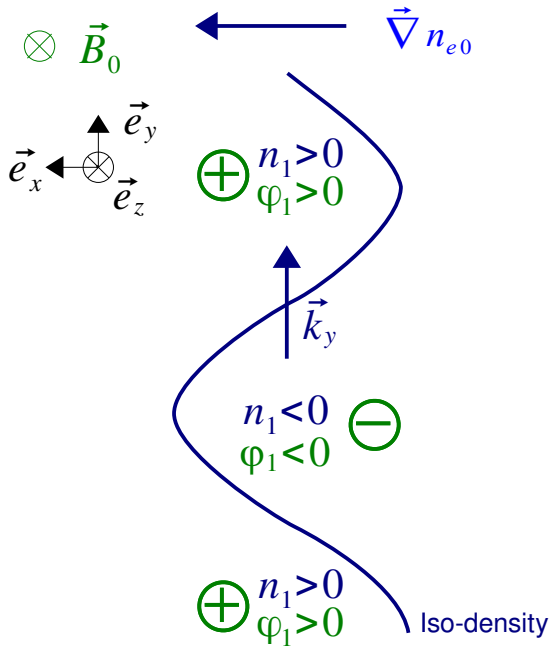
Combining both gives a second relation between n_{e1} and φ_1 : (ion equation)

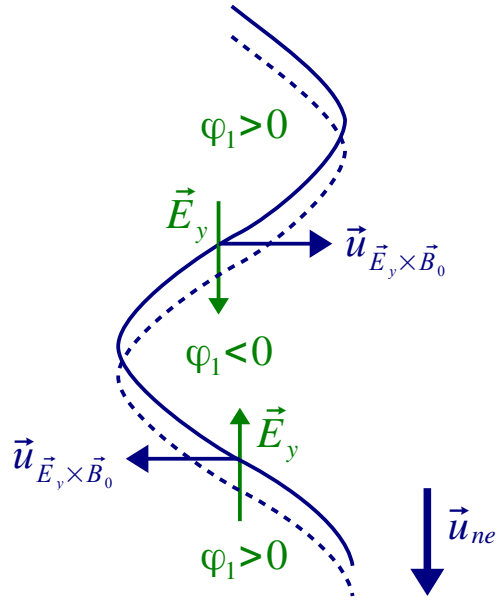
$$\omega\,n_{e1}=-k_y\frac{\varphi_1}{B_{z0}}\partial_x n_{e0}$$

Combining both expressions between n_{e1} and φ_1 (electron and ion equations) so that both parameters compensate.

This gives the dispersion relation:

$$\omega=-k_y\frac{k_BT_e}{q_eB_{z0}}\frac{\partial_x n_{e0}}{n_{e0}}$$





Drift wave : mode propagation

The dispersion relation:

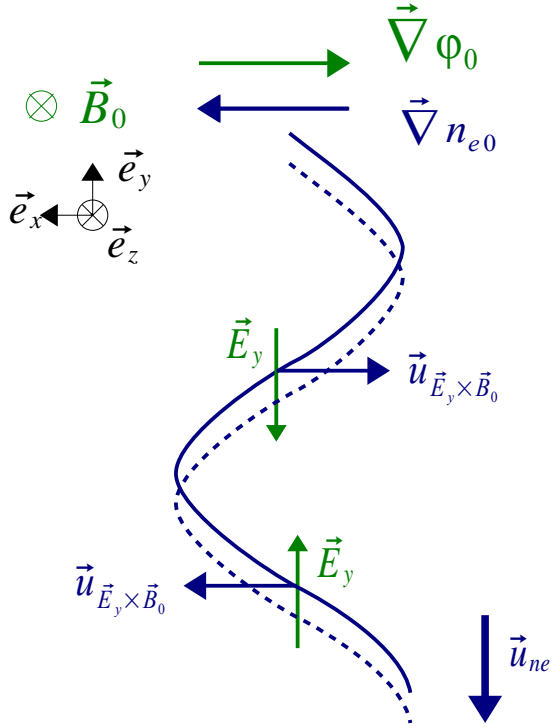
$$\omega = -k_y \frac{k_B T_e}{q_e B_{z0}} \frac{\partial_x n_{e0}}{n_{e0}}$$

The phase velocity ($\vec{u}_\phi = \omega / k_y$) is the diamagnetic velocity.

As a vectors : $\vec{u}_\phi = \vec{u}_{ne} = \frac{k_B T_e}{n_e q_e B_{z0}^2} \vec{\nabla} n_e \times \vec{B}$

If there is an additional global $\vec{\nabla} \varphi_0 \times \vec{B}_0$ drift, the phase velocity is the sum of the diamagnetic velocity and the ExB velocity:

$$\vec{u}_\phi = \vec{u}_{ne} + \vec{u}_{ExB0}$$



Drift wave: condition for mode growth

The dispersion relation is real, it describes only the mode propagation :

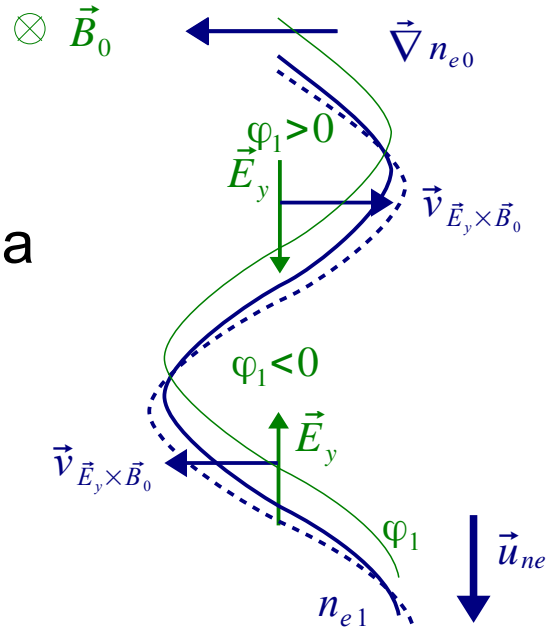
$$\omega = -k_y \frac{k_B T_e}{q_e B_{z0}} \frac{\partial_x n_{e0}}{n_{e0}}$$

Electron equation:

$$\varphi_1 = \frac{k_B T_e}{q_e n_{e0}} n_{e1}$$

We introduce a phase shift between the density and the plasma potential fluctuations:

$$\varphi_1 = \frac{k_B T_e}{q_e n_{e0}} e^{i\phi_{n\varphi}} n_{e1}$$



We keep the ion equation unchanged:

$$\omega n_{e1} = -k_y \frac{\varphi_1}{B_{z0}} \partial_x n_{e0}$$

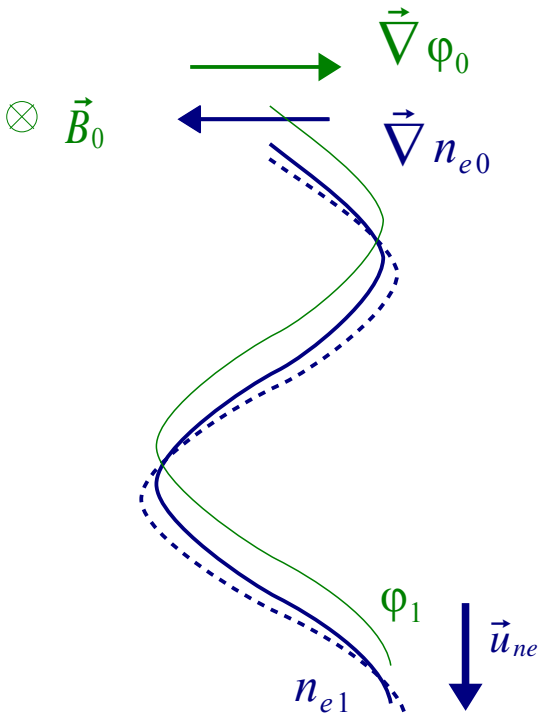
The dispersion equation is modified:

$$\omega = \omega_R + i\gamma = -k_y \frac{k_B T_e}{q_e B_{z0}} \frac{\partial_x n_{e0}}{n_{e0}} e^{i\phi_{n\varphi}}$$

Drift wave: condition for mode growth

The dispersion relation is complex :

$$\omega = \omega_R + i\,\gamma = -k_y \frac{k_B T_e}{q_e B_{z0}} \frac{\partial_x n_{e0}}{n_{e0}} e^{i\phi_{n\varphi}} \qquad \varphi_1 = \frac{k_B T_e}{q_e n_{e0}} e^{i\phi_{n\varphi}} n_{e1}$$



Mode growth rate:

$$\gamma = k_y u_{ne} \sin\phi_{n\varphi}$$

Mode phase velocity :

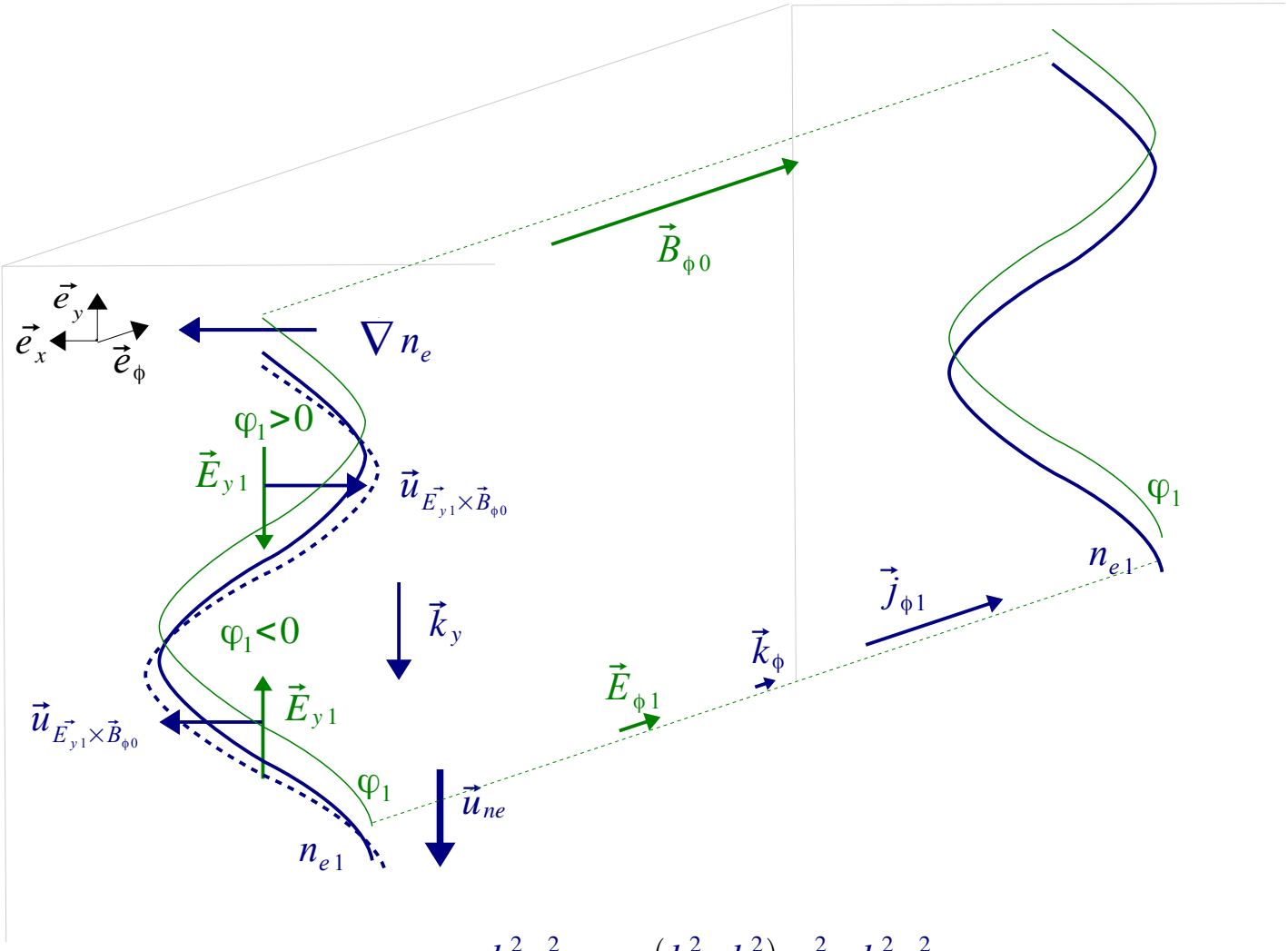
$$u_\phi = \frac{\omega_R}{k_y} = u_{ne} \cos\phi_{n\varphi}$$

Mode phase velocity in presence of potential gradient

$$\vec{u}_\phi = \vec{u}_{ne} \cos\phi_{n\varphi} + \vec{u}_{ExB0}$$

Drift wave: condition for mode growth

3D fluid model is necessary to introduce a mode growth rate

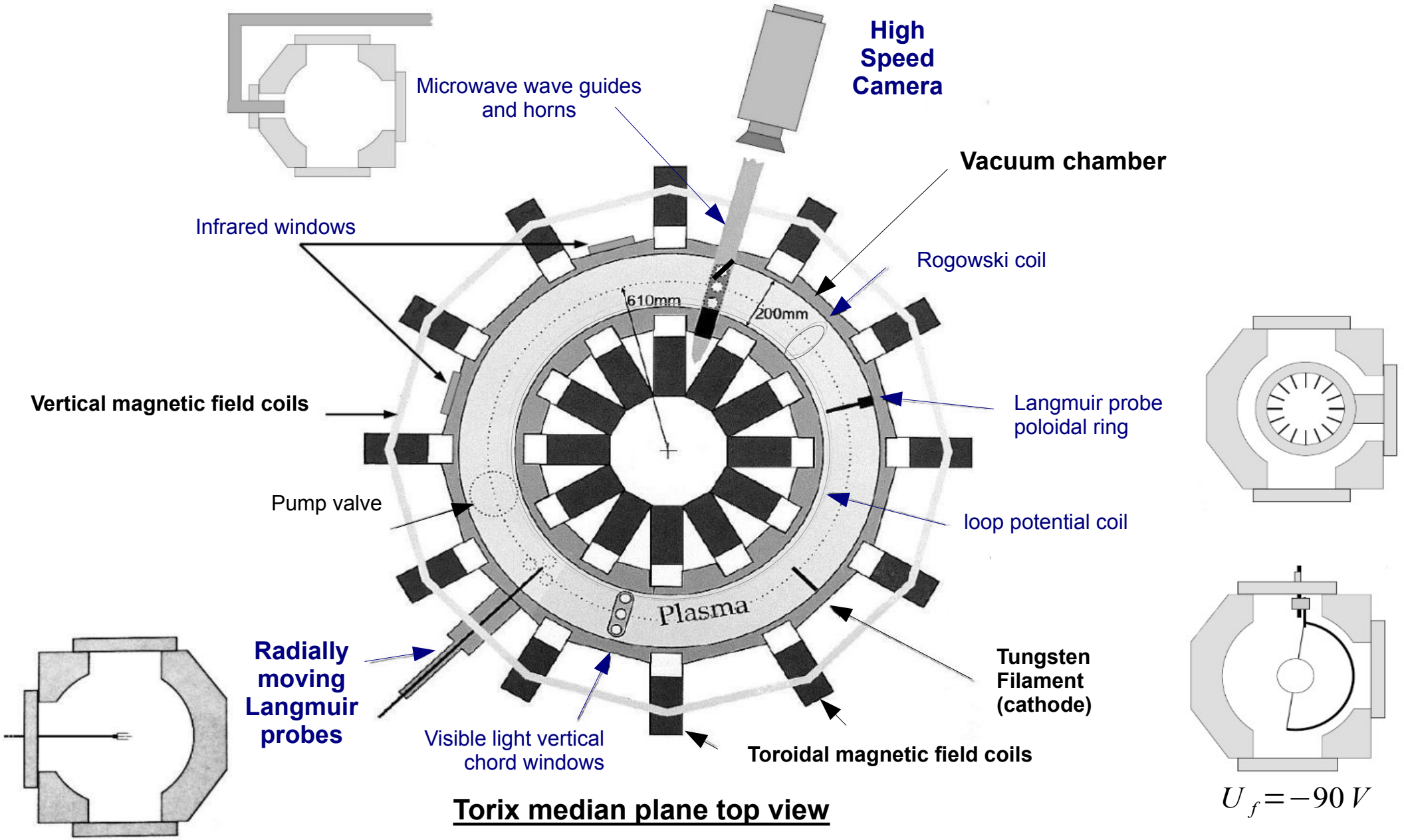


$$\omega - k_y u_{ne} - \frac{k_\phi^2 c_s^2}{\omega} = i \frac{\eta\left(k_x^2 + k_y^2\right)}{\mu_0} \frac{\omega^2 - k_\phi^2 c_s^2}{k_\phi^2 u_A^2}$$

Langmuir Probe Instability Measurements

- Torix diagnostics
- Probe measurements principle
- Ion and Electron sheath

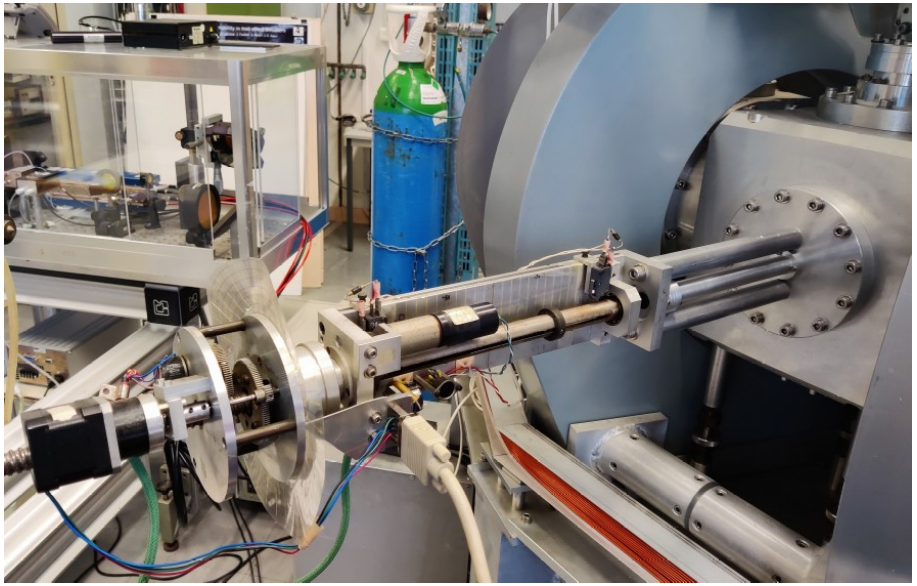
Torix : operation components and diagnostics

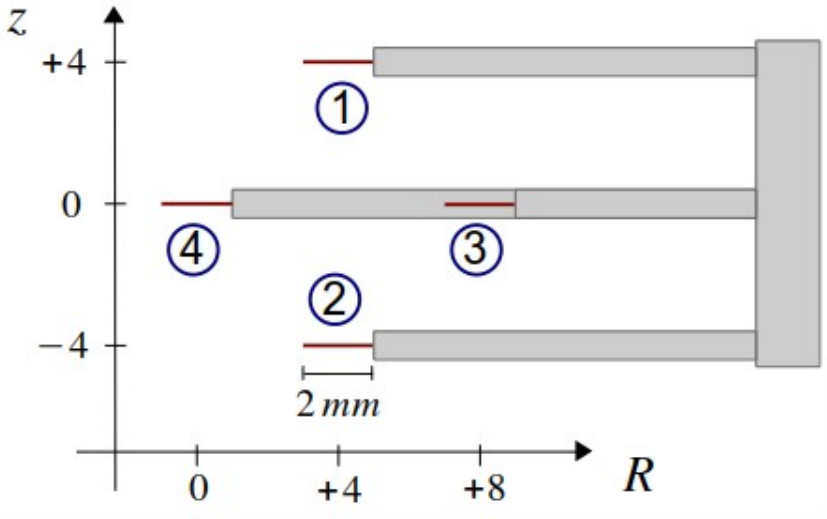


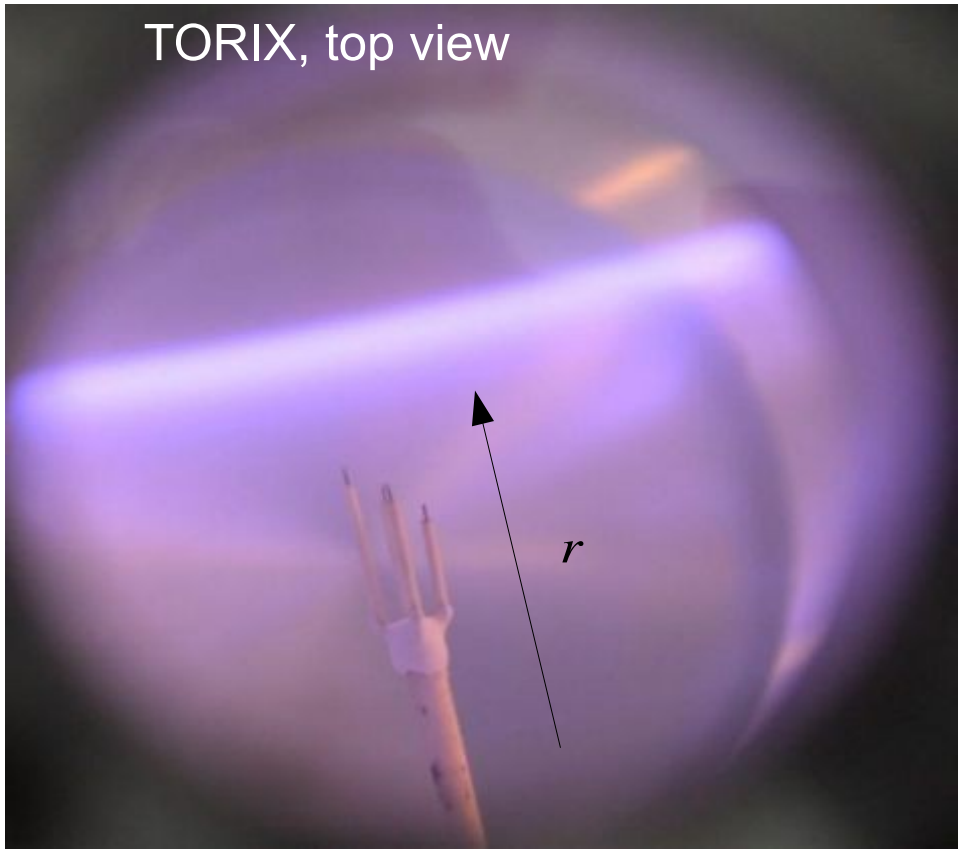
Torix median plane top view

Radially moving Langmuir probes

The radially moving Langmuir probe arm allows plasma parameter radial profile measurements.



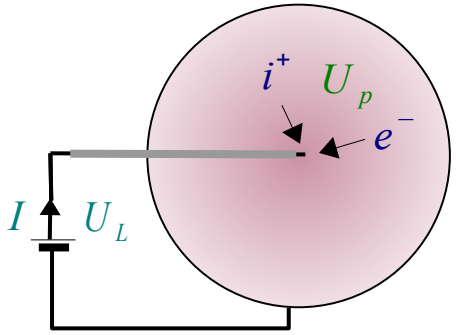




Multiple probes allows to study the instability spatial properties.

Probe principle

Langmuir probe: conductor put inside the plasma.
The probe is polarized at U_L.
The plasma potential is U_p.



Depending on U_L vs U_p, plasma electrons and / or ions are attracted to the probe.

The probe current depends on the electron and ion currents to the probe which depends and the particle **density** and **velocity**, and the probe collection **area**.

Probe sheath (and pre-sheath)

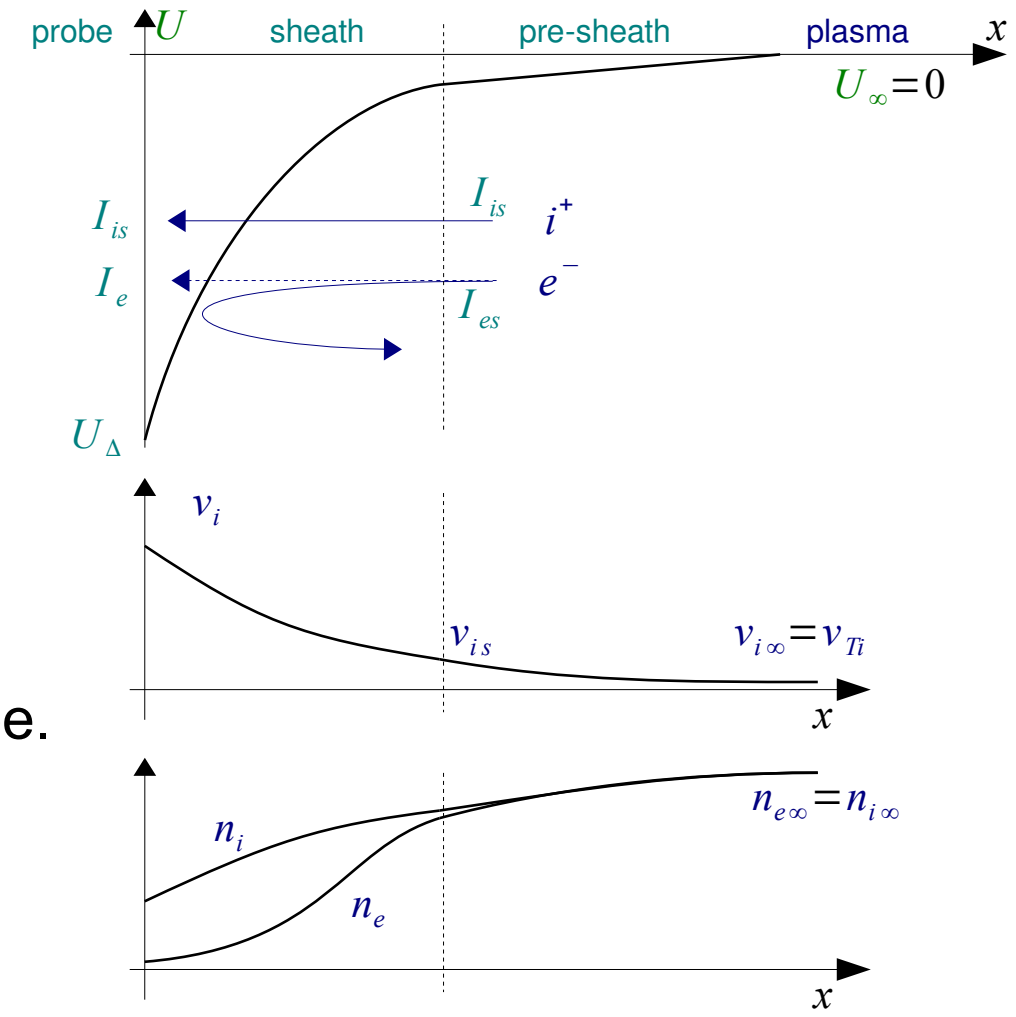
Since the plasma and probe potential are different, there is a space around the probe where the **electric force prevails** upon other forces : this is the **probe sheath**.
Dynamics is described by Poisson equation.

Electron sheath ($U_L > U_p$), electrons are attracted, ions are repulsed.

Ion sheath ($U_L < U_p$), electrons are (partially) repulsed and ions are accelerated to the probe.
Poisson equation have a valid solution if ion velocity at the sheath input is at least:

$$v_{is} = c_s = \sqrt{\frac{k_B Te}{m_i}}$$

There is an initial pre-sheath where other forces are present and the ion are accelerated to reach the Bohm velocity



Electron sheath: electron saturation current $U_L > U_p$

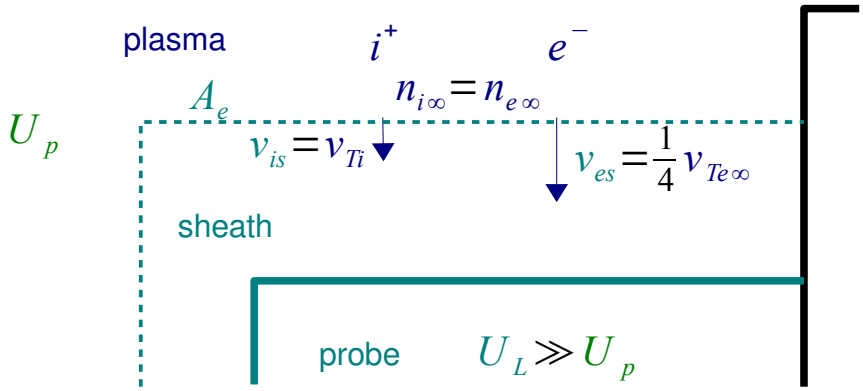
When $U_L > U_p$, all ions are repulsed by the probe.
electrons are attracted by the probe.
There is no pre-sheath
The electron velocity at the probe entrance is 1D thermal velocity:

$$v_{es} = \frac{1}{4} v_{Te\infty} = \frac{1}{4}\sqrt{\frac{8 k_B T_e}{\pi m_e}}$$

The electron saturation current is the particle flux (density * velocity) multiplied by the probe collection area A_e and the particle charge.

$$I_{eS} = q_e n_{es} v_{es} A_e$$

Convention: $I > 0$ for electron



Ion Sheath: ion saturated current $U_L < U_p$

When $U_L < U_p$ ions are slightly accelerated in the pre-sheath. At the sheath entrance:

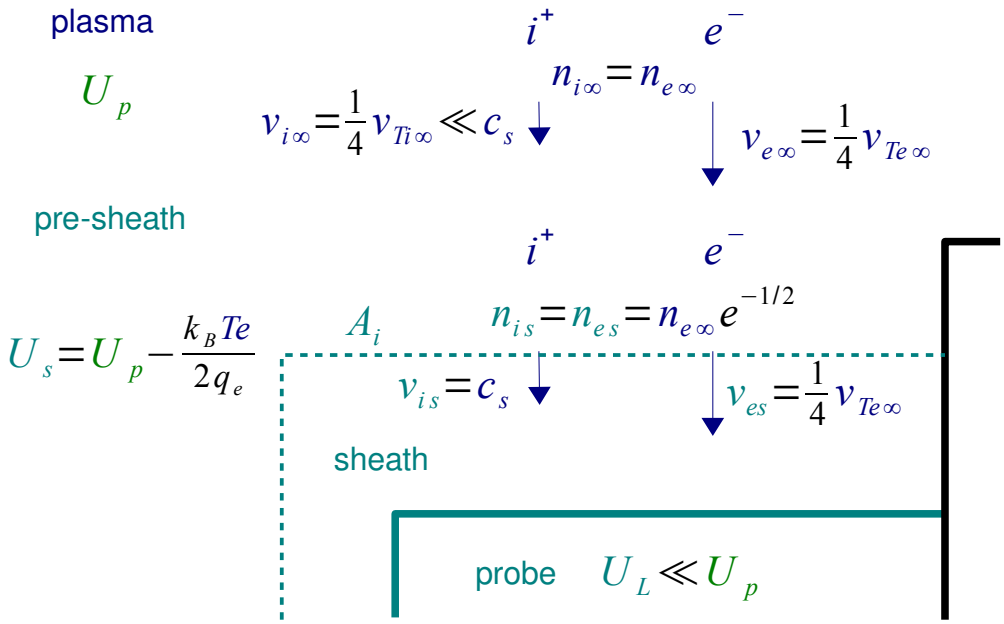
$$v_{is} = c_s = \sqrt{\frac{k_B Te}{m_i}} \qquad n_{is} = n_{es} = n_{e\infty} e^{-1/2}$$

Ion sheath : all ions entering the sheath reach the probe.
The ion saturated current is estimated at the sheath entrance:

$$I_{iS} = -q_e n_{is} v_{is} A_i$$

$$I_{iS} = -q_e n_{e\infty} e^{-1/2} A_i \sqrt{\frac{k_B Te}{m_i}}$$

Since $\sqrt{\frac{m_e}{m_i}} \ll 1$: $I_{iS} \ll I_{es}$



Ion sheath: electron current

For ion sheath, electron that have enough energy go through the sheath potential barrier:

$$U_\Delta = U_L - U_p$$

Electron energy distribution:

$$f_e(E) = \frac{n_{e\infty}}{k_B T_e} e^{\frac{-E}{k_B T_e}}$$

Electron density that can reach the probe:

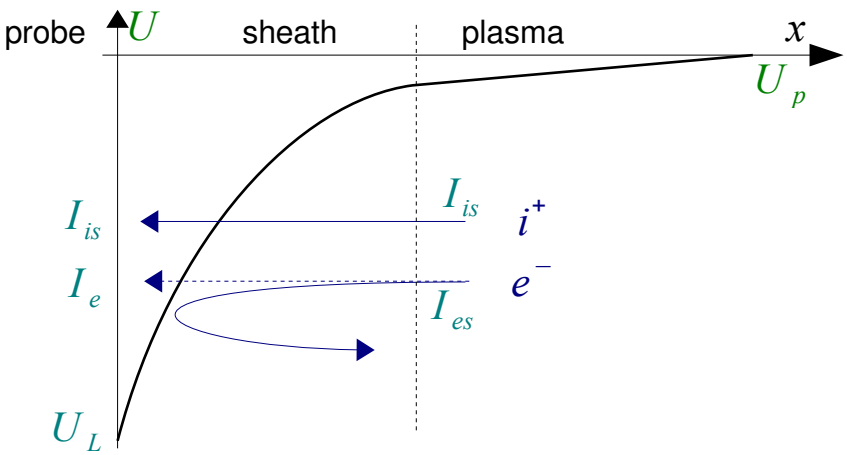
$$n_e = n_{e\infty} e^{\frac{-q_e}{k_B T_e}(U_p - U_L)}$$

Electron current on the probe:

$$I_e(U_L) = I_{eS}(U_L) e^{\frac{-q_e}{k_B T_e}(U_p - U_L)}$$

Total current on the probe:

$$I(U_L) = I_{iS}(U_L) + I_{eS}(U_L) e^{\frac{-q_e}{k_B T_e}(U_p - U_L)}$$



Collection area: geometrical and physical effects

For magnetized plasma, the collection area depends on the magnetic field.
Since charged particles move mainly along the magnetic field direction, the collection is perpendicular to the magnetic field.
If the probe is a cylinder along the magnetic field:

- the collection area is 1st defined by the probe surface:

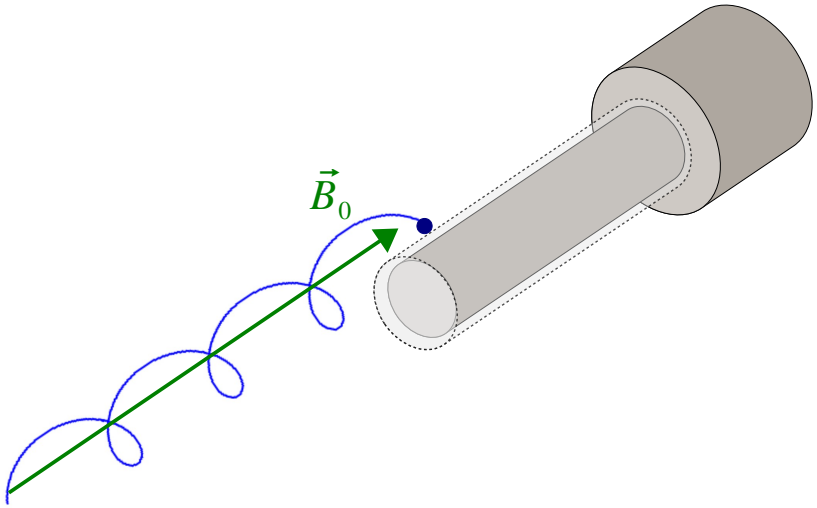
$$A_L = \pi \rho_L^2$$

- the sheath thickness must be added around the probe:

$$A_{si} = \pi \left[\rho_L + e_{is}(U_L) \right]^2$$

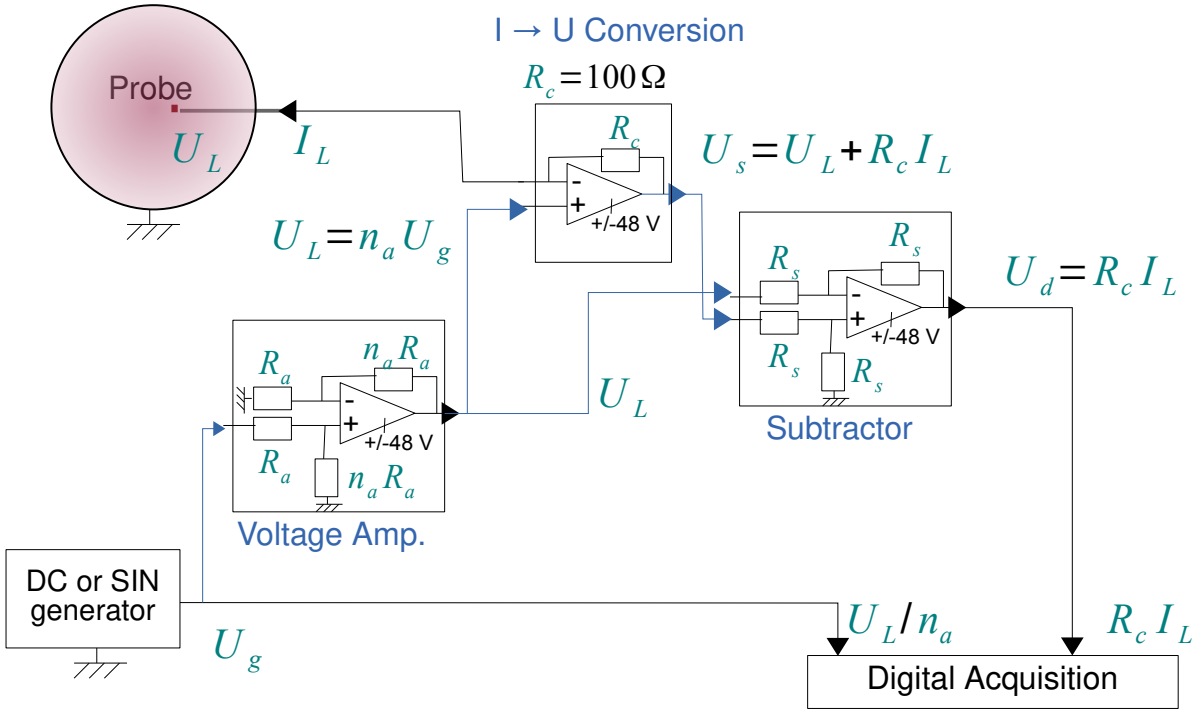
- the particles turn around the magnetic guiding center ; the Larmor radius is also added:

$$A_i = \pi \left[\rho_L + e_{is}(U_L) + \rho_{cTi} \right]^2$$



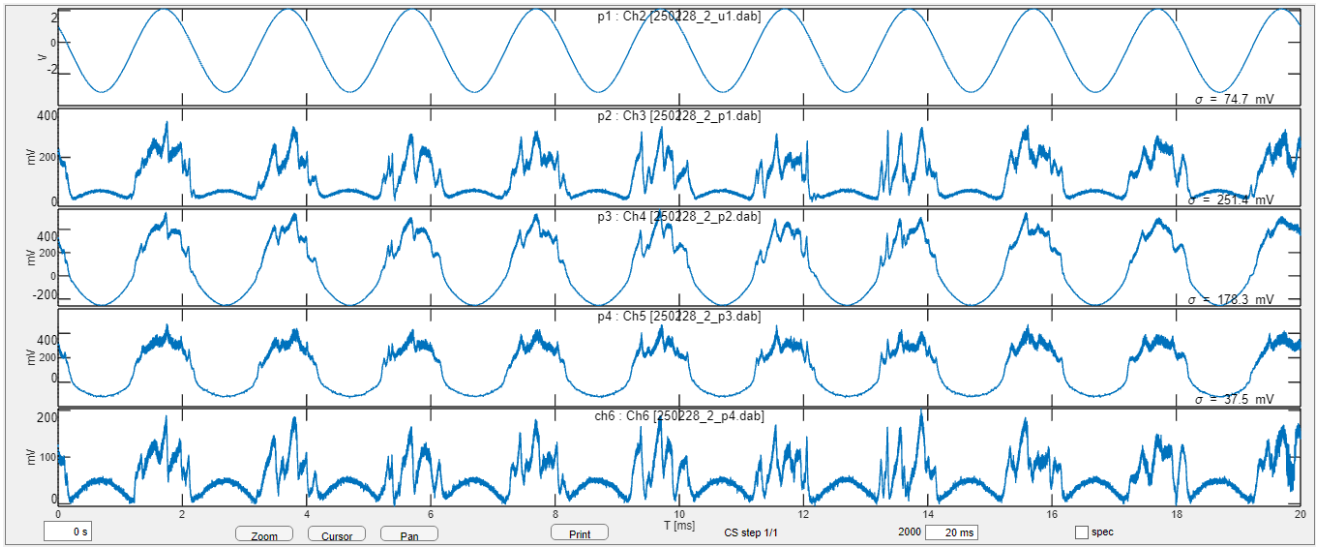
Probe polarization and current acquisition

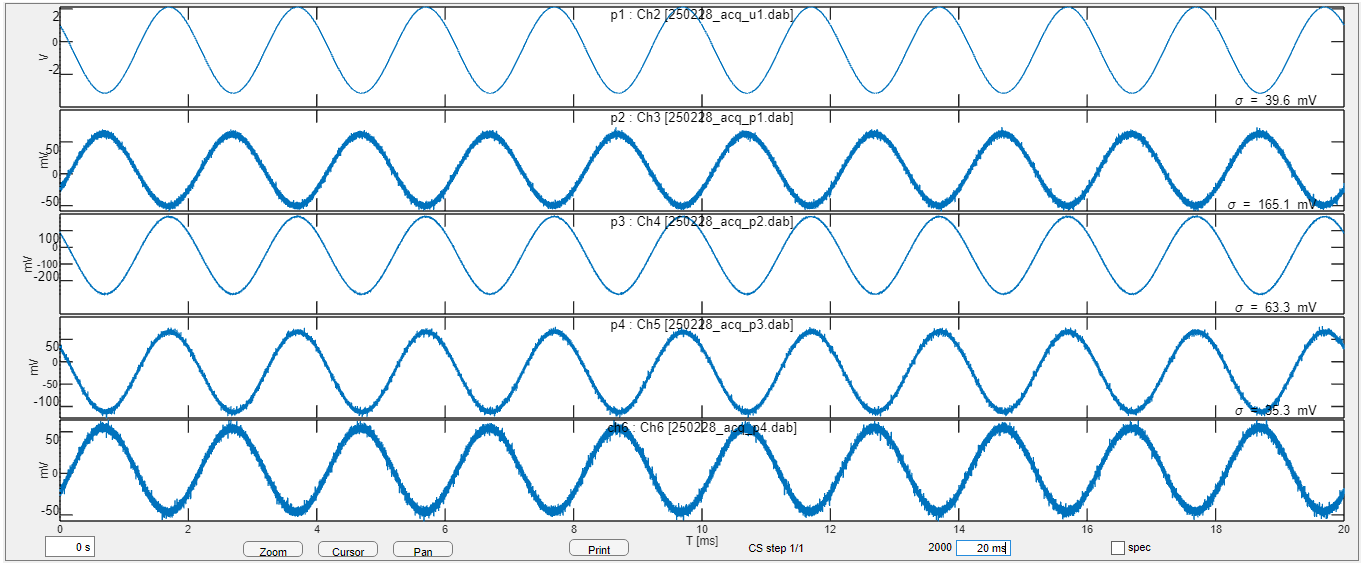
The probe is connected to an op-amp I-U converter :
The probe polarization voltage is applied (-30V … +20 V) the positive op-amp input.
The voltage is supplied by a signal generator through a voltage amplifier.
The probe itself is connected to the negative op-amp input.
The op-amp output is connected to a voltage subtractor (to subtract the polarization voltage)



Probe polarization and current acquisition

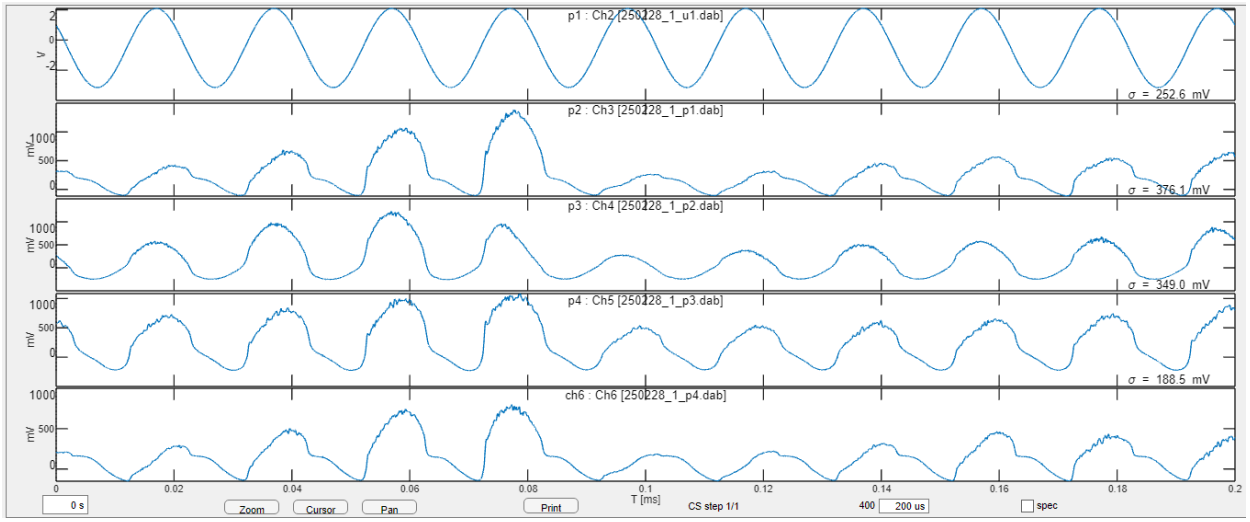
Plasma



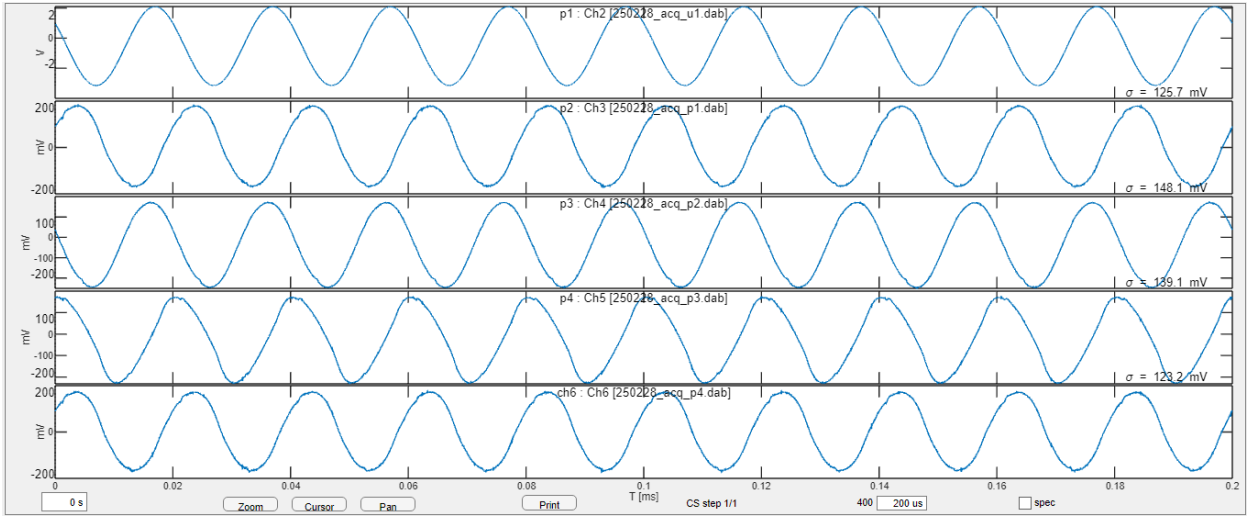


Probe polarization and current acquisition

Probe current measurements are difficult
We have to subtract measurement without plasma from measurement with plasma to compensate the electronic bias in measurement and the capacitive effect of the probe on the circuit.



Probe current measurements with plasma



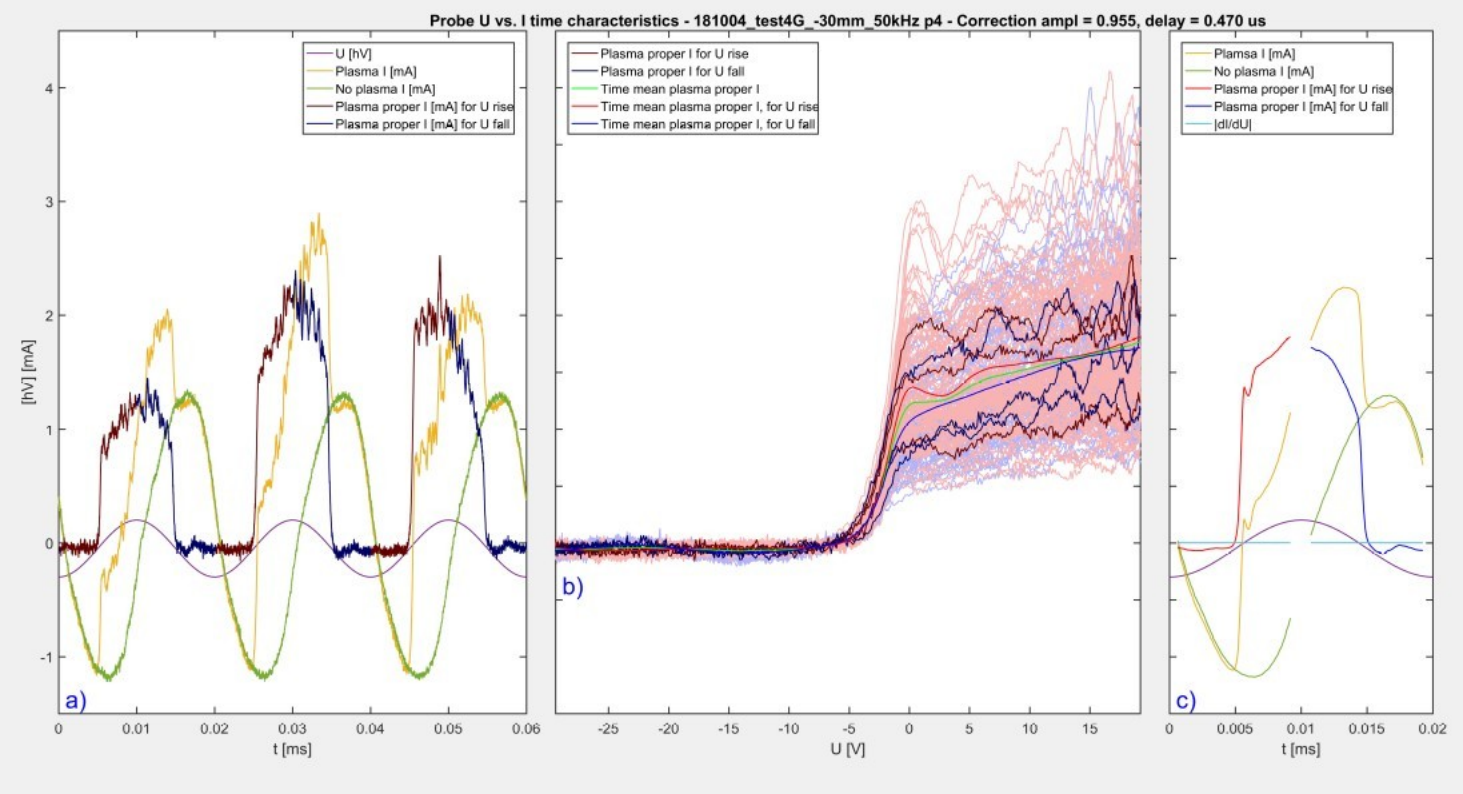
Probe current measurements without plasma

100 kHz probe dynamical measurements

Dynamical measurements are possible.

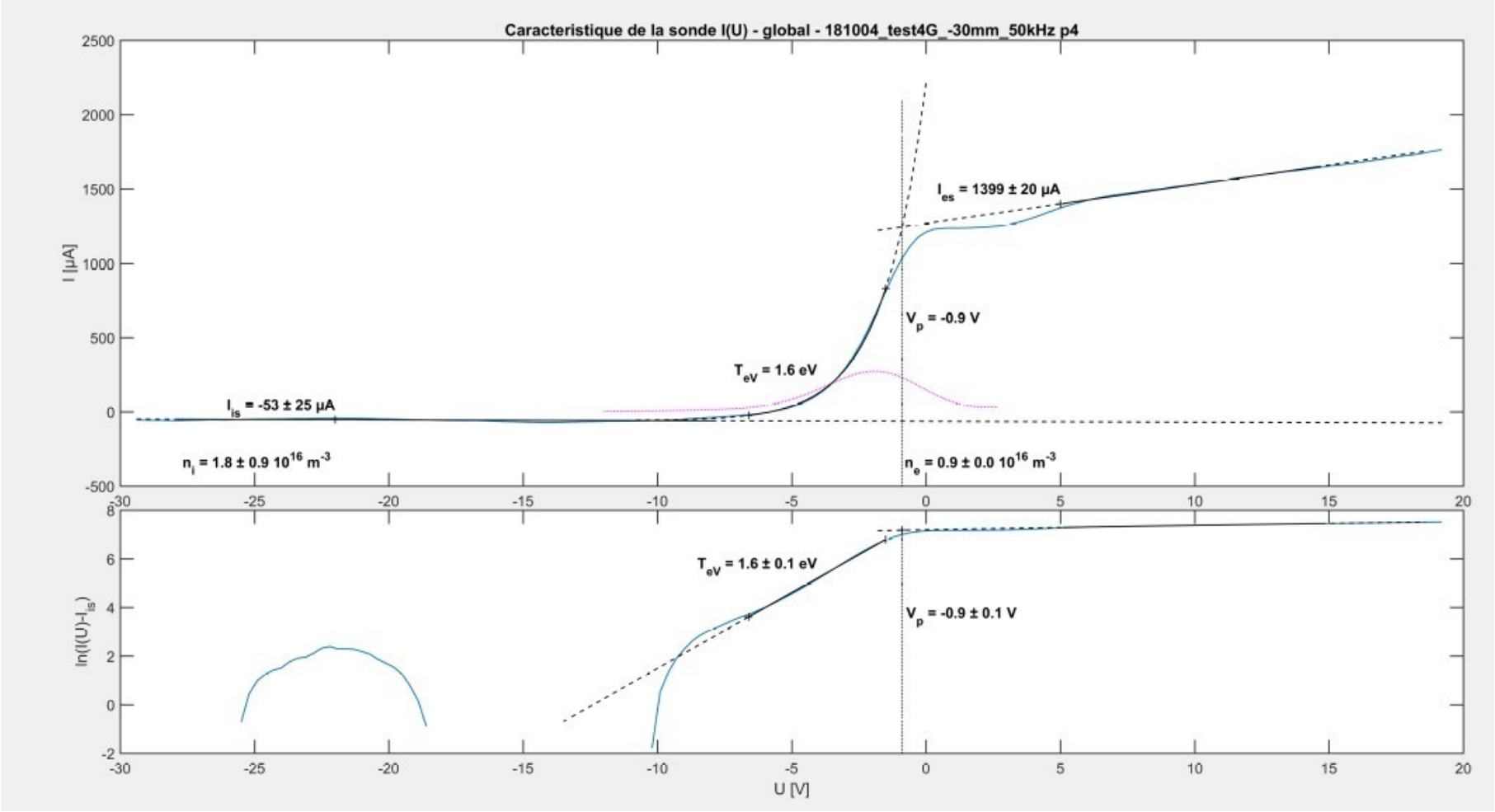
Taking into account the delay between probe generator potential and probe current measurements, the probe characteristics show no hysteresis between probe potential increasing and decreasing phases.

Generator is at 50 kHz, but 2 plasma parameter extractions are possible for each generator period



Typical probe characteristic

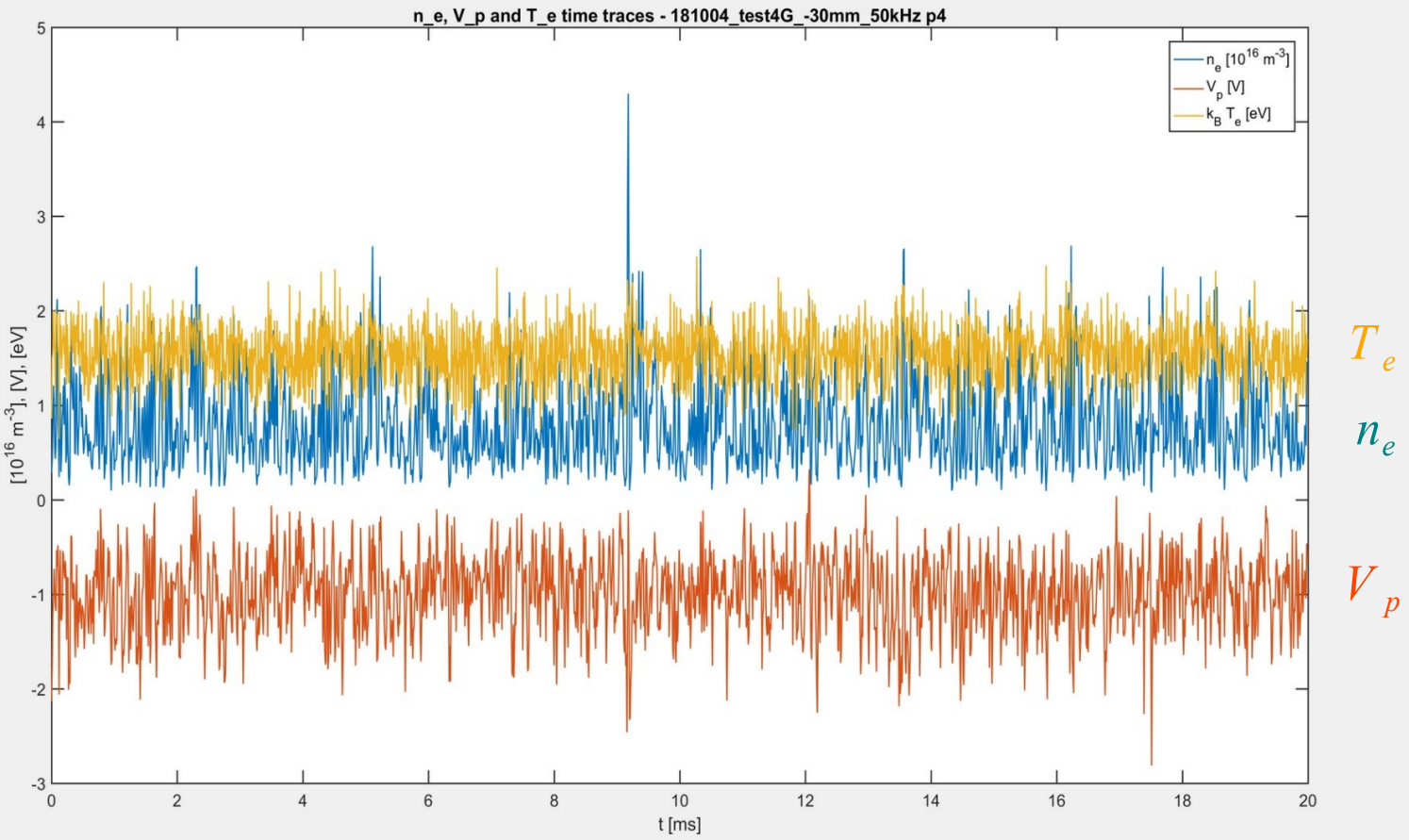
Plasma



100 kHz probe dynamical measurements

Electronic development :
50 kHz dynamic probe polarization and probe current measurements.

Data treatment to extract plasma parameter time signals.



Typical probe characteristic

1D spatial - time Fourier mode: $n_e(x,t)=\tilde{n}_e\cos(k_{y0}y-\omega_0 t)$

Signals for 2 vertically separated positions:

$$n_{e2}(t)=\tilde{n}_e\cos(k_{y0}y_0-\omega_0 t) \qquad n_{e1}(t)=\tilde{n}_e\cos(k_{y0}y_0-\omega_0 t+k_{y0}d)$$

Signal time Fourier transforms:

$$n_{e2}(\omega)\propto\delta(\omega-\omega_0)\tilde{n}_e e^{j(k_{y0}y_0)} \qquad n_{e1}(\omega)\propto\delta(\omega-\omega_0)\tilde{n}_e e^{j(k_{y0}y_0+k_y d)}$$

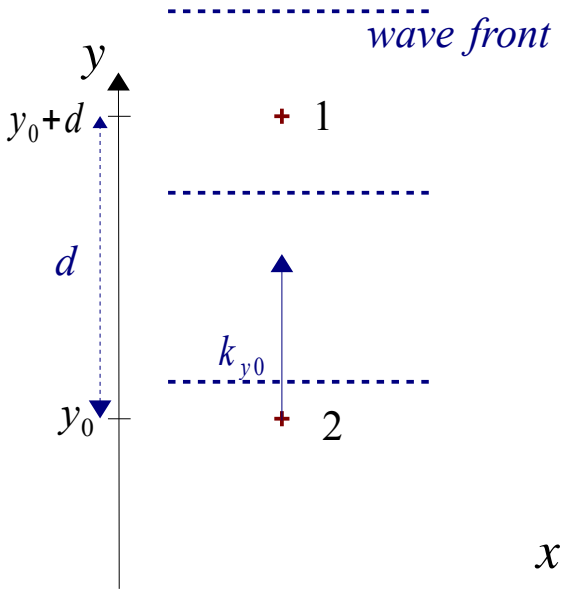
Frequency cross-spectrum: $S_{12}(\omega)\propto n_{e2}(\omega)^* n_{e1}(\omega)$

$$S_{12}(\omega)\propto\delta(\omega-\omega_0)\tilde{n}_e^2 e^{j k_{y0}d}$$

Cross spectrum phase: $\Delta\phi_{y0}=k_{y0}d$

Since $v_{y\phi 0}=\frac{\omega_0}{k_{y0}}$: $\Delta\phi_{y0}=\frac{d}{v_{y\phi 0}}\omega_0$

Delay between signals : $\tau_0=\frac{d}{v_{y\phi 0}}=\frac{k_{y0}d}{\omega_0}$



Annex

Diamagnetic drift

Charged particle velocity distribution with a density or a temperature gradient

Particle position: x

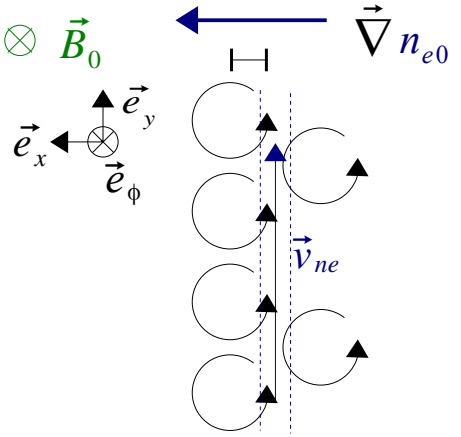
Center guide position: $X = x + \frac{v_y}{\omega_{c\alpha}}$

Density and Temperature depend on X:

$$T_\alpha(X) \qquad n_{g\alpha}(X)$$

Center guide velocity distribution as a function of X:

$$f_{g\alpha}(X,\vec{v}) = \left(\frac{m_\alpha}{2\pi k_B T_\alpha(X)}\right)^{3/2} n_{g\alpha}(X)\, e^{\frac{-m_\alpha v^2}{2k_B T_\alpha(X)}} \qquad \text{(Boltzmann)}$$

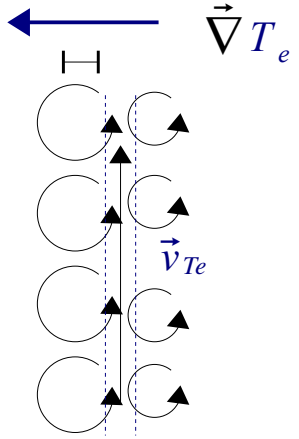


Particle velocity distribution as a function of x:

$$f_{0\alpha}(x,\vec{v}) = f_{g\alpha}\left(x + \frac{v_y}{\omega_{c\alpha}},\vec{v}\right) = f_{g\alpha}(x,\vec{v}) + \partial_X f_{g\alpha}(x,\vec{v})\frac{v_y}{\omega_{c\alpha}}$$

with :

$$\partial_x f_{g\alpha}(x,\vec{v}) = \left[d_X \ln(n_{g\alpha}) + d_X \ln(T_\alpha)\left(\frac{\frac{1}{2}m_\alpha v^2}{k_B T_\alpha} - \frac{3}{2}\right)\right] f_{g\alpha}(x,\vec{v})$$



For density : $n_{0\alpha}(x) = n_{g\alpha}(x)$

For mean velocity : $\langle \vec{v}_{0\alpha}(x) \rangle = \left[d_X \ln(n_{0\alpha}) + d_X \ln(T_\alpha)\right] \frac{k_B T_\alpha}{q_\alpha B} \vec{e}_y$

$$\vec{u}_{P\alpha} = \frac{-\gamma_\alpha k_B T_\alpha}{q_\alpha B_0} \frac{\vec{\nabla} P_\alpha}{P_\alpha} \times \frac{\vec{B}_0}{B_0}$$

Electron sheath: electron saturation current $U_L > U_p$

When $U_L > U_p$, all ions are repulsed by the probe.
electrons are attracted by the probe.
There is no pre-sheath
The electron velocity at the probe entrance is 1D thermal velocity:

$$v_{es} = \frac{1}{4} v_{Te\infty} = \frac{1}{4}\sqrt{\frac{8 k_B T_e}{\pi m_e}}$$

The electron saturation current is the particle flux (density * velocity) multiplied by the probe collection area A_e and the particle charge.

$$I_{eS} = q_e n_{es} v_{es} A_e$$

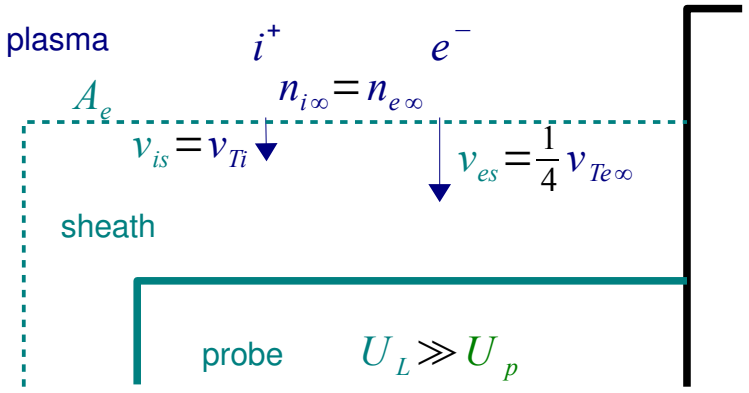
Convention: $I > 0$ for electron

$$I_{eS} = \frac{1}{4} q_e n_{e\infty} A_e \sqrt{\frac{8 k_B T_e}{\pi m_e}}$$

The collection area depends on the sheath thickness:

$$e_{es}(U_\Delta) = 1.26 \left[\frac{q_e U_\Delta}{k_B T_e}\right]^{3/4} \lambda_{De}$$

$$U_\Delta = U_L - U_p$$



Ion Sheath: ion saturated current $U_L < U_p$

Ions are slightly accelerated in the pre-sheath. At the sheath entrance:

$$U_s = U_p - \frac{k_B Te}{2 q_e} \qquad v_{is} = c_s = \sqrt{\frac{k_B Te}{m_i}} \qquad n_{is} = n_{es} = n_{e\infty} e^{-1/2}$$

Ion sheath : all ions entering the sheath reach the probe.
The ion saturated current is estimated at the sheath entrance:

$$I_{iS} = -q_e n_{is} v_{is} A_i$$

$$I_{iS} = -q_e n_{e\infty} e^{-1/2} A_i \sqrt{\frac{k_B Te}{m_i}}$$

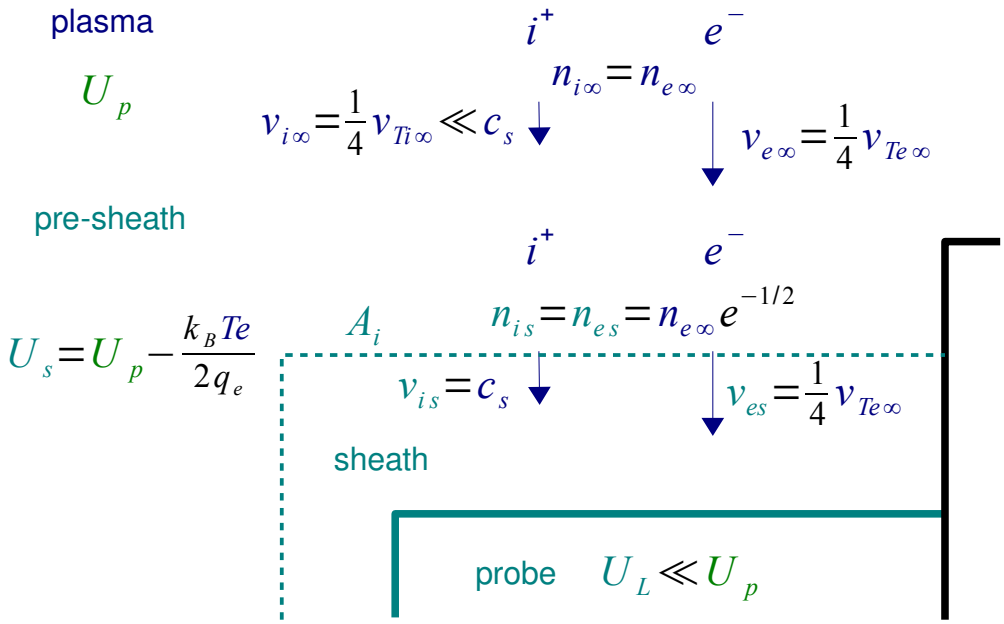
The collection area depends on the sheath thickness:

$$e_{is} = 1.02 \left[\frac{q_e |U_\Delta|}{k_B T_e} \right]^{3/4} \lambda_{De}$$

$$U_\Delta = U_L - U_p$$

The ion current is much smaller than the electron one

$$\sqrt{\frac{m_e}{m_i}} \ll 1 \qquad A_e \sim A_i \qquad I_{iS} \ll I_{es}$$



Finite power signal auto-spectrum and -correlation

Finite power signal:

$$\int_{-\infty}^{+\infty} |s(t)|^2 \, dt = +\infty \qquad \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt < +\infty$$

Signal power:

$$\langle |s(t)|^2 \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt$$

Finite time Fourier transform:

$$\hat{s}_T(\omega) = \int_{-T/2}^{T/2} s(t) \, \mathrm{e}^{-i\omega t} \, dt$$

Frequency auto-spectrum

$$S(\omega) = \lim_{T \to +\infty} \frac{1}{T} |\hat{s}_T(\omega)|^2$$

Signal time auto-correlation:

$$C(\tau) = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} s^*(t) \, s(t+\tau) \, dt \qquad S(f) = \int_{-\infty}^{+\infty} d\tau \, C(\tau) \, \mathrm{e}^{-i 2\pi f \tau}$$

Parseval formula

$$\lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt = C(0) = \int_{-\infty}^{+\infty} S(f) \, df$$

Two signal cross-spectrum and cross-correlation

2 signals: $s_a(t)$ $s_b(t)$

Signal cross-spectrum:

$$S_{ab}(f)=\lim_{T\to+\infty}\frac{1}{T}\hat{s}^*_{aT}(f)\hat{s}_{bT}(f)$$

$$\hat{s}_{aT}(\omega)=\int_{-T/2}^{T/2}s_a(t)\,\mathrm{e}^{-i\omega t}\,dt$$

Signal cross-correlation :

$$C_{ab}(\tau)=\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}s^*_a(t)\,s_b(t+\tau)\,dt$$

$$S_{ab}(f)=\int_{-\infty}^{+\infty}C_{ab}(\tau)\,\mathrm{e}^{-i2\pi f\tau}\,d\tau$$

Normalized signals:

$$\breve{s}_a(t)=\frac{s_a(t)-\langle s_a\rangle}{\sigma_a}$$

$$\sigma_a^2=\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|s_a(t)-\langle s_a\rangle\right|^2dt$$

Auto-spectrum integral and correlation:

$$\int_{-\infty}^{+\infty}S_{\breve{a}\breve{a}}(f)\,df=C_{\breve{a}\breve{a}}(0)=1$$

Cross-correlation and cross-spectrum (Cauchy-Schwarz inequality):

$$-1\le C_{\breve{a}\breve{b}}(\tau)\le 1$$

$$-1\le\int_{-\infty}^{+\infty}S_{\breve{a}\breve{b}}(f)\,df=C_{\breve{a}\breve{b}}(0)\le 1$$

Typical probe characteristic

2D-time Fourier mode:

$$n_e(x,y,t)=\tilde{n}_e\cos(k_x x+k_y y-\omega_0 t)$$

Signals for 2 vertically separated positions:

$$n_{e2}(t)=\tilde{n}_e\cos(k_x x_0+k_y y_0-\omega_0 t) \qquad n_{e1}(t)=\tilde{n}_e\cos(k_x x_0+k_y y_0-\omega_0 t+k_y d)$$

Signal time Fourier transforms:

$$n_{e2}(\omega)\propto\delta(\omega-\omega_0)\tilde{n}_e e^{j(k_x x_0+k_y y_0)} \qquad n_{e1}(\omega)\propto\delta(\omega-\omega_0)\tilde{n}_e e^{j(k_x x_0+k_y y_0+k_y d)}$$

Frequency cross-spectrum: $S_{12}(\omega)\propto n_{e2}(\omega)^* n_{e1}(\omega)$

$$S_{12}(\omega)\propto\delta(\omega-\omega_0)\tilde{n}_e^2 e^{j k_y d}$$

Cross spectrum phase:

$$\Delta\phi_y(\omega_0)=k_y(\omega_0)d$$

Since: $v_{y\phi}(\omega_0)=\frac{\omega_0}{k_y(\omega_0)}$

$$\Delta\phi_y(\omega_0)=\frac{d}{v_{y\phi}(\omega_0)}\omega_0$$

